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CENTRAL FORCE PROBLEMS

Introduction. This is where it all began. Newton's *Mathematical Principles of Natural Philosophy* (1687) was written and published at the insistance of young Enmund Halley (1656–1742) so that the world at large might become acquainted with the general theory of motion that had permitted Newton to assure Halley (on occasion of the comet of 1682) that comets trace elliptical orbits (and that in 1705 permitted Halley to identify that comet—now known as "Halley's comet"—with the comets of 1531 and 1607, and to predict its return in 1758). It was with a remarkably detailed but radically innovative account of the quantum theory of hydrogen that Schrödinger signaled his invention of wave mechanics, a comprehensive quantum theory of motion-in-general. In both instances, description of the general theory was preceded by demonstration of the success of its application to the problem of <u>two bodies in central interaction</u>.[1]

We will be concerned mainly with various aspects of the dynamical problem posed when two bodies interact *via* central conservative forces. But I would like to preface our work with some remarks intended to place that problem in its larger context.

We often yield, as classical physicists, to an unexamined predisposition to suppose that *all* interactions are necessarily 2-body interactions. To describe a system of interacting particles we find it natural to write

$$m_i\ddot{\boldsymbol{x}}_i = \boldsymbol{F}_i + {\sum_j}' \boldsymbol{F}_{ij}$$
$$\boldsymbol{F}_{ij} = \text{force on } i^{\text{th}} \text{ by } j^{\text{th}}$$

[1] It is interesting to note that the *Principia* begins with a series of eight Definitions, of which the last four speak about "centripetal force."

and to write

$$\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji} \quad : \quad \text{“action = reaction”} \tag{1.1}$$

as an expression of Newton's 3$^{\text{rd}}$ Law. But it is entirely possible to contemplate 3-body forces

$$\boldsymbol{F}_{i,jk} = \text{force on } j^{\text{th}} \text{ due to membership in } \{i,j,k\}$$

and to consider it to be a requirement of an enlarged 3$^{\text{rd}}$ Law that[2]

$$\boldsymbol{F}_{i,jk} + \boldsymbol{F}_{j,ki} + \boldsymbol{F}_{k,ij} = \boldsymbol{0} \tag{1.2}$$

For the dynamics of a system of interacting particles to admit of Lagrangian formulation the interparticle forces must be conservative (derivable from a potential). For a 2-particle system we would introduce $U^{\text{interaction}}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ and, to achieve compliance with (1.1), require

$$(\boldsymbol{\nabla}_1 + \boldsymbol{\nabla}_2)U^{\text{interaction}}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{0} \tag{2.1}$$

For a 3-particle system we introduce $U^{\text{interaction}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ and require

$$\begin{aligned} (\boldsymbol{\nabla}_1 + \boldsymbol{\nabla}_2 + \boldsymbol{\nabla}_3)U^{\text{interaction}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) &= \boldsymbol{0} \\ &\vdots \end{aligned} \tag{2.2}$$

The “multi-particle interaction” idea finds natural accommodation within such a scheme, but the 3$^{\text{rd}}$ Law is seen to impose a severe constraint upon the design of the interaction potential.

We expect the interparticle force system to be insensitive to gross translation of the particle population:

$$U^{\text{interaction}}(\boldsymbol{x}_1 + \boldsymbol{a}, \boldsymbol{x}_2 + \boldsymbol{a}, \ldots, \boldsymbol{x}_n + \boldsymbol{a}) = U^{\text{interaction}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$$

This, pretty evidently, requires that the interaction potential depend upon its arguments only through their differences

$$\boldsymbol{r}_{ij} = \boldsymbol{x}_i - \boldsymbol{r}_j$$

of which there are an antisymmetric array:

$$\left.\begin{array}{cccc} \boldsymbol{r}_{12} & \boldsymbol{r}_{13} & \ldots & \boldsymbol{r}_{1n} \\ & \boldsymbol{r}_{23} & \ldots & \boldsymbol{r}_{2n} \\ & & & \vdots \\ & & & \boldsymbol{r}_{n-1,n} \end{array}\right\} \quad : \quad \text{total of } N = \tfrac{1}{2}n(n-1) \text{ such } \boldsymbol{r}\text{'s}$$

[2] Though illustrated here as it pertains to 3-body forces, the idea extends straightforwardly to n-body forces, but the “action/reaction” language seems in that context to lose some of its naturalness. For discussion, see CLASSICAL MECHANICS (1983/84), page 58.

Rotational insensitivity

$$V^{\text{interaction}}(\mathbb{R}\boldsymbol{r}_{12},\mathbb{R}\boldsymbol{r}_{13},\ldots)=V^{\text{interaction}}(\boldsymbol{r}_{12},\boldsymbol{r}_{13},\ldots)$$

pretty evidently requires that the translationally invariant interaction potential depend upon its arguments $\boldsymbol{r}_{ij}$ only through their dot products

$$r_{ij,kl}=\boldsymbol{r}_{ij}\boldsymbol{\cdot}\boldsymbol{r}_{kl}=\boldsymbol{x}_i\boldsymbol{\cdot}\boldsymbol{x}_k-\boldsymbol{x}_i\boldsymbol{\cdot}\boldsymbol{x}_l-\boldsymbol{x}_j\boldsymbol{\cdot}\boldsymbol{x}_k+\boldsymbol{x}_j\boldsymbol{\cdot}\boldsymbol{x}_l$$

of which there is a symmetric array with a total of

$$\tfrac{1}{2}N(N+1)=\tfrac{1}{2}(n^2-n+2)(n-1)n$$

elements.

n	$\frac{1}{2}(n^2-n+2)(n-1)n$
1	0
2	1
3	6
4	21
5	55
6	120

TABLE 1: *Number of arguments that can appear in a translationally and rotationally invariant potential that describes n-body interaction.*

In the case $n=2$ one has $U(r_{12,12})=U([\boldsymbol{r}_1-\boldsymbol{r}_2]\boldsymbol{\cdot}[\boldsymbol{r}_1-\boldsymbol{r}_2])$ giving

$$\begin{aligned}\boldsymbol{\nabla}_1U&=+2U'(r_{12,12})\cdot[\boldsymbol{r}_1-\boldsymbol{r}_2]\\ \boldsymbol{\nabla}_2U&=-2U'(r_{12,12})\cdot[\boldsymbol{r}_1-\boldsymbol{r}_2]\end{aligned}$$

whence

$$\boldsymbol{\nabla}_1U+\boldsymbol{\nabla}_2U=\boldsymbol{0}$$

We conclude that *conservative interaction forces, if translationally/rotationally invariant, are automatically central, automatically conform to Newton's* 3^{rd} *Law*. In the case $n=3$ one has $U(r_{12,12},r_{12,13},r_{12,23},r_{13,13},r_{13,23},r_{23,23})$ and—consigning the computational labor to *Mathematica*—finds that compliance with the (extended formulation of) the 3^{rd} Law is again automatic:

$$\boldsymbol{\nabla}_1U+\boldsymbol{\nabla}_2U+\boldsymbol{\nabla}_3U=\boldsymbol{0}$$

I am satisfied, even in the absence of explicit proof, that a similar result holds in every order.

Celestial circumstance presented Newton with several instances (sun/comet, sun/planet, earth/moon) of what he reasonably construed to be instances of the 2-body problem, though it was obvious that they became so by dismissing

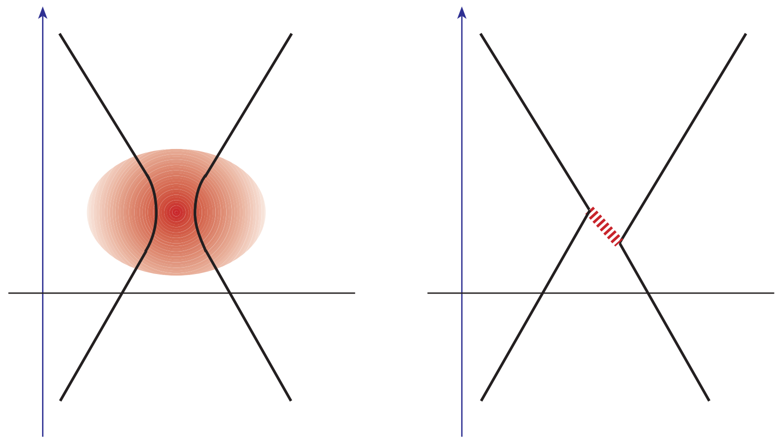

FIGURE 1: *At left: a classical scattering process into which two particles enter, and from which (after some action-at-a-distance has gone on) two particles depart. Bound interaction can be thought of as an endless sequence of scattering processes. At right: mediated interaction, as contemplated in relativistic theories... which include the theory of elementary particles. Primitive scattering events are local and have not four legs but three: one particle enters and two emerge, else two enter and one emerges. In the figure the time axis runs* ↑.

spectator bodies as "irrelevant" (at least in leading approximation[3]). That only 2-body interactions contribute to the dynamics of celestial many-body systems is a proposition enshrined in Newton's **universal law of gravitational interaction**, which became the model for many-body interactions of *all* types. The implicit claim here, in the language we have adopted, is that the interaction potentials encountered in Nature possess the specialized structure

$$U^{\text{interaction}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) = \sum_{\text{pairs}} U_{ij}(r_{ij})$$

where $r_{ij} \equiv \sqrt{r_{ij,ij}} = \sqrt{\boldsymbol{r}_{ij} \cdot \boldsymbol{r}_{ij}}$ and from which arguments of the form $r_{ij,kl}$ (three or more indices distinct) are absent. To create a many-body celestial mechanics Newton would, in effect, have us set

$$U_{ij}(r_{ij}) = -Gm_i m_j \frac{1}{r_{ij}}$$

whence

$$\boldsymbol{F}_{ij} = -\boldsymbol{\nabla}_i U_{ij} = -Gm_i m_j \frac{1}{r_{ij}^2} \hat{\boldsymbol{r}}_{ij}$$

[3] Halley, in constructing his predicted date (1758) of cometary return, took into account a close encounter with Saturn.

where, it will be recalled, $\boldsymbol{r}_{ij} \equiv \boldsymbol{x}_i - \boldsymbol{x}_j = -\boldsymbol{r}_{ji}$ is directed $\boldsymbol{x}_i \longleftarrow \boldsymbol{x}_j$ and $\boldsymbol{F}_{ij}$ refers to the force *impressed upon* m_i *by* m_j.

Though classical mechanics provides a well developed "theory of collisions," the central force problem—our present concern—was conceived by Newton to involve **instantaneous action at a distance**, a concept to which many of his contemporaries (especially in Europe) took philosophical exception,[4] and concerning his use of which Newton himself appears to have been defensively apologetic: he insisted that he "did not philosophize," was content simply to calculate. . . when the pertinence of his calculations was beyond dispute. But in the 20th Century action-at-a-distance ran afoul—conceptually, if not under ordinary circumstances practically—of the Principle of Relativity, with its enforced abandonment of the concept of distant simultaneity. Physicists found themselves *forced* to adopt the view that *all interaction is local, and all remote action mediated*, whether by fields or by real/virtual particles. See FIGURE 1 and its caption for remarks concerning this major conceptual shift.

1. Reduction of the 2-body problem to the equivalent 1-body problem. Suppose it to be the case that particles m_1 and m_2 are subject to no forces except for the conservative central forces which they exert upon each other. Proceeding in reference to a Cartesian inertial frame,[5] we write

$$\left.\begin{aligned} m_1\ddot{\boldsymbol{x}}_1 &= -\boldsymbol{\nabla}_1 U\big(\sqrt{(\boldsymbol{x}_1-\boldsymbol{x}_2)\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}\,\big) \\ m_2\ddot{\boldsymbol{x}}_2 &= -\boldsymbol{\nabla}_2 U\big(\sqrt{(\boldsymbol{x}_1-\boldsymbol{x}_2)\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}\,\big) \end{aligned}\right\} \tag{3}$$

A change of variables renders this system of equations much more amenable to solution. Writing

$$\begin{aligned} m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2 &= (m_1+m_2)\boldsymbol{X} \\ \boldsymbol{x}_1 - \quad \boldsymbol{x}_2 &= \qquad\qquad \boldsymbol{R} \end{aligned} \tag{4.1}$$

we have

$$\begin{aligned} \boldsymbol{x}_1 &= \boldsymbol{X} + \tfrac{m_2}{m_1+m_2}\boldsymbol{R} \\ \boldsymbol{x}_2 &= \boldsymbol{X} - \tfrac{m_1}{m_1+m_2}\boldsymbol{R} \end{aligned} \quad \text{whence} \quad \begin{aligned} \boldsymbol{r}_1 &= +\tfrac{m_2}{m_1+m_2}\boldsymbol{R} \\ \boldsymbol{r}_2 &= -\tfrac{m_1}{m_1+m_2}\boldsymbol{R} \end{aligned} \tag{4.2}$$

and the equations (3) decouple:

$$M\ddot{\boldsymbol{X}} = \boldsymbol{0} \tag{5.1}$$

$$\ddot{\boldsymbol{R}} = -\big(\tfrac{1}{m_1} + \tfrac{1}{m_2}\big)\boldsymbol{\nabla} U(R) \tag{5.2}$$

Equation (5.1) says simply that in the absence of externally impressed forces

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[4] They held to the so-called "Principle of Contiguity," according to which objects interact only by touching.

[5] It would be impossible to talk about the dynamics of two bodies in a world that contains *only* the two bodies. The subtle presence of a universe full of spectator bodies appears to be necessary to lend physical meaning to the inertial frame concept. . . as was emphasized first by E. Mach, and later by A. S. Eddington.

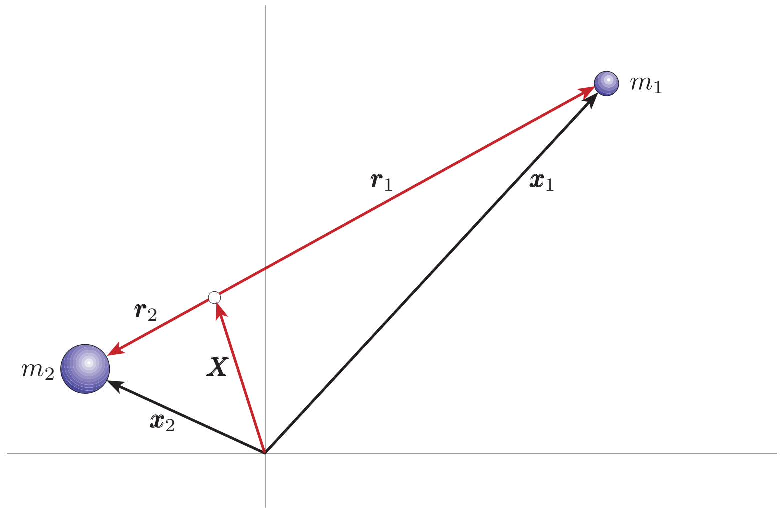

FIGURE 2: *Coordinates $\boldsymbol{x}_i$ position the particles m_i with respect to an inertial frame, $\boldsymbol{X}$ locates the center of mass of the 2-body system, vectors $\boldsymbol{r}_i$ describe particle position relative to the center of mass.*

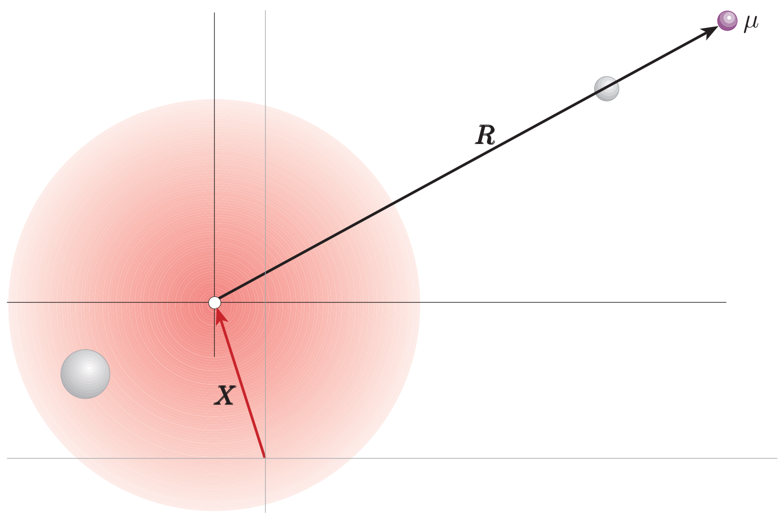

FIGURE 3: *Representation of the equivalent one-body system.*

the motion of the center of mass is unaccelerated. Equation (5.2) says that the vector $\boldsymbol{R} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2 = \boldsymbol{r}_1 - \boldsymbol{r}_2$ moves as though it referred to the motion of a particle of "reduced mass" μ

$$\tfrac{1}{\mu} = \tfrac{1}{m_1} + \tfrac{1}{m_2} \quad \Longleftrightarrow \quad \mu = \tfrac{m_1 m_2}{m_1 + m_2} \tag{6}$$

in an impressed central force field

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{R}) &= -\boldsymbol{\nabla} U(R)\\ &= -U'(R)\hat{\boldsymbol{R}}\\ &\Downarrow\\ &= -\frac{Gm_1m_2}{R^2}\hat{\boldsymbol{R}} \quad \text{in the gravitational case}\end{aligned} \tag{7}$$

FIGURE 3 illustrates the sense in which the one-body problem posed by (5.2) is "equivalent" to the two-body problem from which it sprang.

DIGRESSION:[6] Suppose vectors $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\}$ mark (relative to an inertial frame) the instantaneous positions of a population $\{m_1, m_2, \ldots, m_N\}$ of $N > 2$ particles. We have good reason—rooted in the (generalized) 3rd Law[7]—to expect the center of mass

$$\boldsymbol{X} \equiv \tfrac{1}{M}\sum_i m_i\boldsymbol{x}_i \quad : \quad M \equiv \sum_i m_i$$

to retain its utility, and know that in many contexts the relative coordinates $\boldsymbol{r}_i \equiv \boldsymbol{x}_i - \boldsymbol{X}$ do too. But we cannot adopt $\{\boldsymbol{X}, \boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N\}$ as independent variables, for the system has only $3N$ (not $3N+3$) degrees of freedom, and the $\boldsymbol{r}_i$ are subject at all times to the constraint $\sum_i m_i\boldsymbol{r}_i = \mathbf{0}$. To drop one (which one?) of the $\boldsymbol{r}_i$ would lead to a formalism less symmetric that the physics it would describe. It becomes therefore natural to ask: *Can the procedure* (4) *that served so well in the case $N = 2$ be adapted to cases $N > 2$?* The answer is: Yes, but not so advantageously as one might have anticipated or desired.

Reading from FIGURE 4B, we have

$$\begin{aligned}\boldsymbol{R}_2 &= \boldsymbol{x}_1 - \boldsymbol{x}_2\\ \boldsymbol{R}_3 &= \tfrac{1}{m_1+m_2}(m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2) - \boldsymbol{x}_3\\ \boldsymbol{R}_4 &= \tfrac{1}{m_1+m_2+m_3}(m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2 + m_3\boldsymbol{x}_3) - \boldsymbol{x}_4\\ \boldsymbol{X} &= \tfrac{1}{m_1+m_2+m_3+m_4}(m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2 + m_3\boldsymbol{x}_3 + m_4\boldsymbol{x}_4)\end{aligned}$$

from which it follows algebraically that

$$\left.\begin{aligned}\boldsymbol{x}_4 &= \boldsymbol{X} - \tfrac{m_1+m_2+m_3}{m_1+m_2+m_3+m_4}\boldsymbol{R}_4 &&= \boldsymbol{X} + \boldsymbol{r}_4\\ \boldsymbol{x}_3 &= \boldsymbol{X} + \tfrac{m_4}{m_1+m_2+m_3+m_4}\boldsymbol{R}_4 - \tfrac{m_1+m_2}{m_1+m_2+m_3}\boldsymbol{R}_3 &&= \boldsymbol{X} + \boldsymbol{r}_3\\ \boldsymbol{x}_2 &= \boldsymbol{X} + \tfrac{m_4}{m_1+m_2+m_3+m_4}\boldsymbol{R}_4 + \tfrac{m_3}{m_1+m_2+m_3}\boldsymbol{R}_3 - \tfrac{m_1}{m_1+m_2}\boldsymbol{R}_2 &&= \boldsymbol{X} + \boldsymbol{r}_2\\ \boldsymbol{x}_1 &= \boldsymbol{X} + \tfrac{m_4}{m_1+m_2+m_3+m_4}\boldsymbol{R}_4 + \tfrac{m_3}{m_1+m_2+m_3}\boldsymbol{R}_3 + \tfrac{m_2}{m_1+m_2}\boldsymbol{R}_2 &&= \boldsymbol{X} + \boldsymbol{r}_1\end{aligned}\right\} \tag{8}$$

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[6] This material has been adapted from §5 in "Constraint problem posed by the center of mass concept in non-relativistic classical/quantum mechanics" (1998).

[7] See again §1 in Chapter 2.

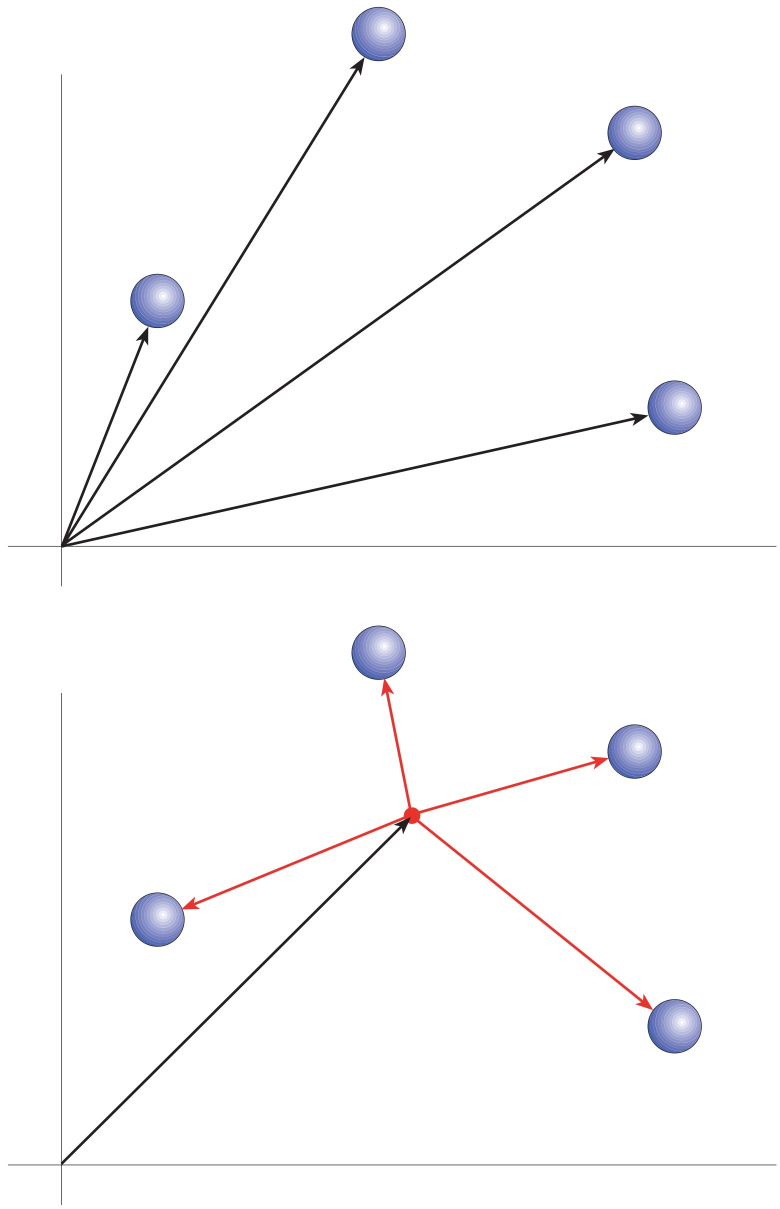

FIGURE 4A: *Shown above*: *the vectors* $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$ *that serve to describe the instantaneous positions of* $\{m_1, m_2, m_3, m_4\}$ *relative to an inertial frame. Shown below*: *the vector* $\boldsymbol{X}$ *that marks the position of the center of mass* • *and the vectors* $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4\}$ *that serve—redundantly—to describe position relative to the center of mass.*

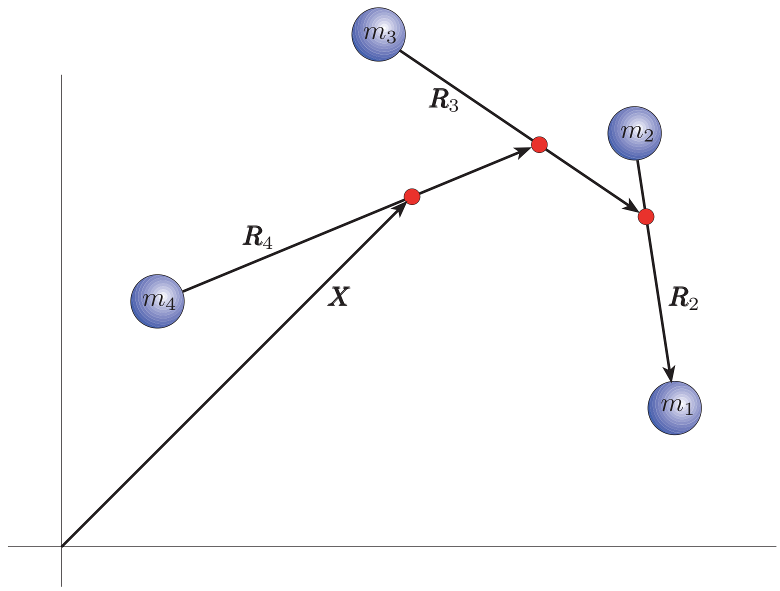

FIGURE 4B: *"Calder construction" of a system of Jacobi vectors. Here*

$\boldsymbol{R}_2$ *proceeds* $m_2 \longrightarrow m_1$
$\boldsymbol{R}_3$ *proceeds* $m_3 \longrightarrow$ *center of mass* • *of* $\{m_1, m_2\}$
$\boldsymbol{R}_4$ *proceeds* $m_4 \longrightarrow$ *center of mass* • *of* $\{m_1, m_2, m_3\}$
$\vdots$
$\boldsymbol{X}$ *marks the center of mass of the entire population*

Alternative Jacobi systems would result if the particle names were permuted.

What he have in (8) is the description of a change of variables[8]

$$\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\} \longrightarrow \{\boldsymbol{X}, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_N\}$$

that serves to render compliance with $\sum m_i \boldsymbol{r}_i = \boldsymbol{0}$ *automatic*. Introduction of the $\boldsymbol{R}$-variables has permitted us to avoid the "discriminatory asymmetry" of $\boldsymbol{r}_1 = -\frac{1}{m_1}\{m_2\boldsymbol{r}_2 + \cdots + m_N\boldsymbol{r}_N\}$, but at cost of introducing an asymmetry of a new sort: a population of N masses can be "mobilized" in $N!$ distinct ways; to select one is to reject the others, and to introduce hierarchical order where (typically) none is present in the physics.

[8] Through presented in the case $N = 4$, it is clear how one would extend (8) to higher order. To pull back to order $N = 3$ one has only to strike the first equation and then to set $m_4 = 0$ in the equations that remain.

So far as concerns the dynamical aspects of that physics, we find (with major assistance by *Mathematica*) that

$$
\begin{aligned}
&\tfrac{1}{2}\big\{m_1\dot{\boldsymbol{x}}_1\cdot\dot{\boldsymbol{x}}_1+m_2\dot{\boldsymbol{x}}_2\cdot\dot{\boldsymbol{x}}_2\big\}\\
&\qquad=\tfrac{1}{2}\big(m_1+m_2\big)\dot{\boldsymbol{X}}\cdot\dot{\boldsymbol{X}}+\tfrac{1}{2}\mu_2\dot{\boldsymbol{R}}_2\cdot\dot{\boldsymbol{R}}_2\\
&\tfrac{1}{2}\big\{m_1\dot{\boldsymbol{x}}_1\cdot\dot{\boldsymbol{x}}_1+m_2\dot{\boldsymbol{x}}_2\cdot\dot{\boldsymbol{x}}_2+m_3\dot{\boldsymbol{x}}_3\cdot\dot{\boldsymbol{x}}_3\big\}\\
&\qquad=\tfrac{1}{2}\big(m_1+m_2+m_3\big)\dot{\boldsymbol{X}}\cdot\dot{\boldsymbol{X}}+\tfrac{1}{2}\mu_2\dot{\boldsymbol{R}}_2\cdot\dot{\boldsymbol{R}}_2+\tfrac{1}{2}\mu_3\dot{\boldsymbol{R}}_3\cdot\dot{\boldsymbol{R}}_3\\
&\tfrac{1}{2}\big\{m_1\dot{\boldsymbol{x}}_1\cdot\dot{\boldsymbol{x}}_1+m_2\dot{\boldsymbol{x}}_2\cdot\dot{\boldsymbol{x}}_2+m_3\dot{\boldsymbol{x}}_3\cdot\dot{\boldsymbol{x}}_3+m_4\dot{\boldsymbol{x}}_4\cdot\dot{\boldsymbol{x}}_4\big\}\\
&\qquad=\tfrac{1}{2}\big(m_1+m_2+m_3+m_4\big)\dot{\boldsymbol{X}}\cdot\dot{\boldsymbol{X}}+\tfrac{1}{2}\mu_2\dot{\boldsymbol{R}}_2\cdot\dot{\boldsymbol{R}}_2+\tfrac{1}{2}\mu_3\dot{\boldsymbol{R}}_3\cdot\dot{\boldsymbol{R}}_3+\tfrac{1}{2}\mu_4\dot{\boldsymbol{R}}_4\cdot\dot{\boldsymbol{R}}_4\\
&\qquad\vdots
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_2 &\equiv \Big[\tfrac{1}{m_1}+\tfrac{1}{m_2}\Big]^{-1}=\tfrac{m_1m_2}{m_1+m_2}\\
\mu_3 &\equiv \Big[\tfrac{1}{\mu_2}+\tfrac{1}{m_3}\Big]^{-1}=\tfrac{(m_1+m_2)m_3}{m_1+m_2+m_3}\\
\mu_4 &\equiv \Big[\tfrac{1}{\mu_3}+\tfrac{1}{m_4}\Big]^{-1}=\tfrac{(m_1+m_2+m_3)m_4}{m_1+m_2+m_3+m_4}\\
&\ \vdots
\end{aligned}
\tag{9}
$$

serve to generalize the notion of "reduced mass." The fact that *no cross terms appear* when kinetic energy is described in terms of $\big\{\boldsymbol{X},\boldsymbol{R}_2,\dots,\boldsymbol{R}_N\big\}$ variables is—though familiar in the case $N=2$—somewhat surprising in the general case. I look to the underlying mechanism, as illustrated in the case $N=3$: we have

$$
\begin{pmatrix}\boldsymbol{r}_1\\ \boldsymbol{r}_2\\ \boldsymbol{r}_3\end{pmatrix}=\mathbb{M}\begin{pmatrix}\boldsymbol{R}_2\\ \boldsymbol{R}_3\end{pmatrix}
$$

$$
\mathbb{M}\equiv\begin{pmatrix}+\frac{m_2}{m_1+m_2} & +\frac{m_3}{m_1+m_2+m_3}\\[4pt] -\frac{m_1}{m_1+m_2} & +\frac{m_3}{m_1+m_2+m_3}\\[4pt] 0 & -\frac{m_1+m_2}{m_1+m_2+m_3}\end{pmatrix}\ :\ \text{Note that } \mathbb{M} \text{ is } 3\times 2
$$

The claim—verified by *Mathematica*—is that

$$
\mathbb{M}^{\mathsf{T}}\begin{pmatrix}m_1&0&0\\0&m_2&0\\0&0&m_3\end{pmatrix}\mathbb{M}=\begin{pmatrix}\mu_2&0\\0&\mu_3\end{pmatrix}
$$

But while the $\boldsymbol{R}$-variables are well-adapted to the description of kinetic energy, we see from[9]

$$\begin{aligned}
\boldsymbol{r}_1 - \boldsymbol{r}_2 &= \boldsymbol{R}_2\\
\boldsymbol{r}_1 - \boldsymbol{r}_3 &= \tfrac{m_2}{m_1+m_2}\boldsymbol{R}_2 + \boldsymbol{R}_3\\
\boldsymbol{r}_1 - \boldsymbol{r}_4 &= \tfrac{m_2}{m_1+m_2}\boldsymbol{R}_2 + \tfrac{m_3}{m_1+m_2+m_3}\boldsymbol{R}_3 + \boldsymbol{R}_4\\
\boldsymbol{r}_2 - \boldsymbol{r}_3 &= -\tfrac{m_1}{m_1+m_2}\boldsymbol{R}_2 + \boldsymbol{R}_3\\
\boldsymbol{r}_2 - \boldsymbol{r}_4 &= -\tfrac{m_1}{m_1+m_2}\boldsymbol{R}_2 + \tfrac{m_3}{m_1+m_2+m_3}\boldsymbol{R}_3 + \boldsymbol{R}_4\\
\boldsymbol{r}_3 - \boldsymbol{r}_4 &= -\tfrac{m_1+m_2}{m_1+m_2+m_3}\boldsymbol{R}_3 + \boldsymbol{R}_4
\end{aligned}$$

that $\boldsymbol{R}$-variables are (except in the case $N = 2$) not particularly well-adapted to the description of (the distances which separate the masses, whence to the description of) central 2-body interactive forces. In the case of the graviational 3-body problem we now find ourselves led to write

$$\begin{aligned}
U = -G\Big\{&m_1m_2\big[R_2^2\big]^{-\frac{1}{2}} + m_1m_3\big[\big(\tfrac{m_2}{m_1+m_2}\big)^2R_2^2 + \tfrac{m_2}{m_1+m_2}\boldsymbol{R}_2\cdot\boldsymbol{R}_3 + R_3^2\big]^{-\frac{1}{2}}\\
&+ m_2m_3\big[\big(\tfrac{m_1}{m_1+m_2}\big)^2R_2^2 - \tfrac{m_1}{m_1+m_2}\boldsymbol{R}_2\cdot\boldsymbol{R}_3 + R_3^2\big]^{-\frac{1}{2}}\Big\}\\
\downarrow\quad&\\
= -Gm_1m_2/&\sqrt{\boldsymbol{R}_2\cdot\boldsymbol{R}_2} \quad \text{when } m_3 \text{ is extinguished}
\end{aligned}$$

which provides one indication of why it is that the 2-body problem is so much easier than the 3-body problem, but at the same time suggests that the variables $\boldsymbol{R}_2$ and $\boldsymbol{R}_3$ may be of real use in this physical application. As, apparently, they turn out to be: consulting A. E. Roy's *Orbital Motion* (1978), I discover (see his §5.11.3) that $\boldsymbol{r} \equiv -\boldsymbol{R}_2$ and $\boldsymbol{\rho} \equiv -\boldsymbol{R}_3$ were introduced by Jacobi and Lagrange, and are known to celestial mechanics as "Jacobian coordinates." For an interesting recent application, and modern references, see R. G. Littlejohn & M. Reinseh, "Gauge fields in the separation of rotations and internal motions in the n-body problem," RMP **69**, 213 (1997).

It is interesting to note that the pretty idea from which this discussion has proceeded (FIGURE 4B) was elevated to the status of (literally) fine art by Alexander Calder (1898–1976), the American sculptor celebrated for his invention of the "mobile."

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[9] The following equations can be computed algebraically from (19). But they can also—and more quickly—be read off directly from FIGURE 4B: to compute $\boldsymbol{r}_i - \boldsymbol{r}_j$ one starts at ⓙ and walks along the figure to ⓘ, taking signs to reflect whether one proceeds prograde or retrograde along a given leg, and (when one enters/exits at the "fulcrum" $\circ$ of a leg) taking a fractional factor which conforms to the "teeter-totter principle"

$$\text{factional factor} = \frac{\text{mass to the rear of that leg}}{\text{total mass associated with that leg}}$$

A little practice shows better than any explanation how the procedure works, and how wonderfully efficient it is.

2. Mechanics of the reduced system: motion in a central force field. We study the system

$$L(\dot{\boldsymbol{r}},\boldsymbol{r}) = \tfrac{1}{2}\mu\,\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} - U(r) \tag{10}$$

where I have written $\boldsymbol{r}$ for the vector that came to us (FIGURE 3) as $\boldsymbol{R} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$. Equivalently

$$H(\boldsymbol{p},\boldsymbol{r}) = \tfrac{1}{2\mu}\,\boldsymbol{p}\cdot\boldsymbol{p} + U(r) \tag{11}$$

where

$$\boldsymbol{p} \equiv \partial L/\partial\dot{\boldsymbol{r}} = \mu\dot{\boldsymbol{r}}$$

The Lagrange equations read $\mu\ddot{\boldsymbol{r}} + \boldsymbol{\nabla}U = 0$ or (compare (5.2))

$$\mu\ddot{\boldsymbol{r}} = -\tfrac{1}{r}\,U'(r)\,\boldsymbol{r} \tag{12}$$

which in the Hamiltonian formalism are rendered

$$\left.\begin{aligned} \dot{\boldsymbol{r}} &= \tfrac{1}{\mu}\boldsymbol{p} \\ \dot{\boldsymbol{p}} &= -\tfrac{1}{r}\,U'(r)\,\boldsymbol{r} \end{aligned}\right\} \tag{13}$$

From the time-independence of the Lagrangian it follows (by Noether's theorem) that energy is conserved

$$E = \tfrac{1}{2}\mu\,\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} + U(r) \quad \text{is a constant of the motion} \tag{14}$$

while from the manifest rotational invariance of the Lagrangian it follows (on those same grounds) that angular momentum[10] is conserved

$$\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p} \quad \text{is a vectorial constant of the motion} \tag{15}$$

We can anticipate that once values have been assigned to E and $\boldsymbol{L}$ the general solution $\boldsymbol{r}(t;E,\boldsymbol{L})$ of the equation(s) of motion (12) will contain two adjustable parameters. In Hamiltonian mechanics (14) reduces to the triviality $[H,H\,] = 0$ while (15) becomes

$$[H,\boldsymbol{L}] = \boldsymbol{0} \tag{16}$$

Since $\boldsymbol{L}$ stands $\perp$ to the plane defined by $\boldsymbol{r}$ and $\boldsymbol{p}$,[11] and since also $\boldsymbol{L}$ is invariant, it follows that the vectors $\boldsymbol{r}(t)$ are confined to a plane—the *orbital plane*, normal to $\boldsymbol{L}$, that contains the force center as a distinguished point. The existence of an orbital plane can be understood physically on grounds that—because the force is central—the particle never experiences a force that would pull it out of the plane defined by $\{\boldsymbol{r}(0),\dot{\boldsymbol{r}}(0)\}$.

---

[10] What is here called "angular momentum" would, in its original 2-body context, be called "intrinsic angular momentum" or "spin" to distinguish it from the "orbital angular momentum" $M\boldsymbol{X}\times\dot{\boldsymbol{X}}$: the familiar distinction here is between "angular momentum *of* the center of mass" and "angular momentum *about* the center of mass."

[11] The case $\boldsymbol{r} \parallel \boldsymbol{p}$ is clearly exceptional: definition of the plane becomes ambituous, and $\boldsymbol{L} = \boldsymbol{0}$.

Reorient the reference frame so as to achieve

$$\boldsymbol{L} = \begin{pmatrix} 0 \\ 0 \\ \ell \end{pmatrix}$$

and install polar coordinates on the orbital plane:

$$\begin{aligned} r_1 &= r\cos\theta \\ r_2 &= r\sin\theta \end{aligned}$$

We then have

$$\begin{aligned} L &= T - U \\ &= \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \end{aligned} \tag{17}$$

Time-independence implies conservation of

$$E = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) \tag{18}$$

while θ-independence implies conservation of $p_\theta \equiv \partial L/\partial\dot{\theta} = \mu r^2\dot{\theta}$. But from

$$\begin{aligned} L_3 = r_1p_2 - r_2p_1 &= \mu\big[r\cos\theta(\dot{r}\sin\theta + r\dot{\theta}\cos\theta) - r\sin\theta(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\big] \\ &= \mu r^2\dot{\theta} \end{aligned} \tag{19}$$

we see that p_θ, L_3 and ℓ are just different names for the same thing. From (18) we obtain

$$\dot{r} = \sqrt{\tfrac{2}{\mu}\big[E - U(r)\big] - r^2\dot{\theta}^2}$$

which by

$$\dot{\theta} = \ell/\mu r^2 \tag{20}$$

becomes

$$= \sqrt{\tfrac{2}{\mu}\big[E - U(r) - \tfrac{\ell^2}{2\mu r^2}\big]} \tag{21}$$

This places us in position once again to describe a "time of flight"

$$t_{r_0\to r} = \int_{r_0}^{r} \frac{1}{\sqrt{\tfrac{2}{\mu}\big[E - U(r) - \tfrac{\ell^2}{2\mu r^2}\big]}}\,dr \tag{22}$$

which by functional inversion (if it could be performed) would supply $r(t)$. Moreover

$$\frac{d\theta}{dr} = \frac{\dot{\theta}}{\dot{r}} = \frac{\ell/\mu r^2}{\sqrt{\tfrac{2}{\mu}\big[E - U(r) - \tfrac{\ell^2}{2\mu r^2}\big]}}$$

which provides this "angular advance" formula

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{\ell/\mu r^2}{\sqrt{\frac{2}{\mu}\big[E - U(r) - \frac{\ell^2}{2\mu r^2}\big]}}\,dr \tag{23}$$

But again, a (possibly intractable) functional inversion stands between this and the $r(\theta; E, \ell)$ with which we would expect to describe an orbit in polar coordinates.

At (21) the problem of two particles (masses m_1 and m_2) moving interactively in 3-space has been reduced to the problem of one particle (mass μ) moving on the positive half of the r-line in the presence of the effective potential

$$U_\ell(r) = U(r) + \frac{\ell^2}{2\mu r^2} \tag{24}$$

—has been reduced, in short, to the problem posed by the effective Lagrangian

$$L_\ell = \tfrac{1}{2}\mu\dot{r}^2 - U_\ell(r)$$

The beauty of the situation is that in 1-dimensional problems it is possible to gain powerful insight from the simplest of diagramatic arguments. We will rehearse some of those, as they relate to the Kepler problem and other special cases, in later sections. In the meantime, see FIGURE 5.

Bound orbits arise when the values of E and ℓ are such that r is trapped between turning points r_{min} and r_{max}. In such cases one has

$$\begin{aligned}\Delta\theta &\equiv \text{angular advance per radial period}\\ &= 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{\ell/\mu r^2}{\sqrt{\frac{2}{\mu}\big[E - U(r) - \frac{\ell^2}{2\mu r^2}\big]}}\,dr\end{aligned} \tag{25}$$

with consequences illustrated in FIGURE 6. An orbit will close (and motion along it be periodic) if there exist integers m and n such that

$$m\Delta\theta = n2\pi$$

Many potentials give rise to *some* closed orbits, but it is the upshot of **Bertrand's theorem**[12] that only two power-law potentials

$$\begin{aligned} U(r) &= kr^2 &&: \quad \text{isotropic oscillator}\\ U(r) &= -k/r &&: \quad \text{Kepler problem}\end{aligned}$$

have the property that *every* bound orbit closes (and in both of those cases closure occurs after a single circuit.

[12] I return to this topic in §7, but in the meantime, see §3.6 and Appendix A in the 2nd edition (1980) of H. Goldstein's *Classical Mechanics*. . Or see §2.3.3 in J. V. José & E. Saletan, *Classical Dynamics*: *A Contemporary Approach* (1998).

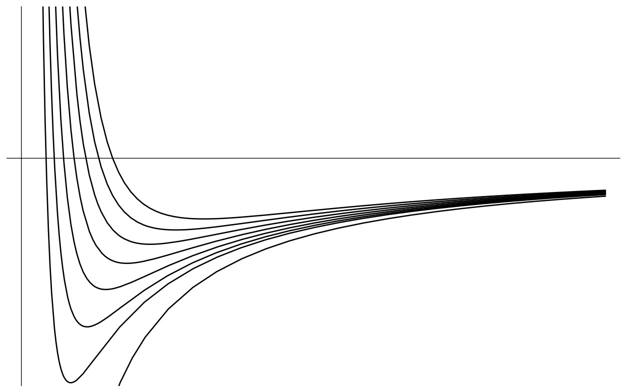

FIGURE 5: *Graphs of* $U_\ell(r)$*, shown here for ascending values of angular momentum* ℓ *in the Keplerian case* $U(r) = -k/r$ *(lowest curve). When* $E < 0$ *the orbit is bounded by turning points at* $r_{\min}$ *and* $r_{\max}$*. When* $r_{\min} = r_{\max}$ *the orbit is necessarily circular (pursued with constant* r*) and the energy is least-possible for the specified value of* ℓ*. When* $E \geqslant 0$ *the orbit has only one turning radius:* $r_{\max}$ *ceases to exist, and the physics of bound states becomes the physics of scattering states. The radius* $r_{\min}$ *of closest approach is* ℓ*-dependent, decreasing as* ℓ *decreases.*

Circular orbits (which, of course, *always*—after a single circuit—close upon themselves) occur only when the values of E and ℓ are so coordinated as to achieve $r_{\min} = r_{\max}$ (call their mutual value r_0). The energy E is then least possible for the specified angular momentum ℓ (and *vice versa*). For a circular orbit one has (as an instance of $T = \frac{1}{2}I\omega^2 = L^2/2I$)

$$\text{kinetic energy } T = \frac{\ell^2}{2\mu r_0^2}$$

but the relation of T to $E = T + U(r_0)$ obviously varies from case to case. But here the **virial theorem**[13] comes to our rescue, for it is an implication of that pretty proposition that if $U(r) = kr^n$ then on a circular orbit $T = \frac{n}{2}U(r_0)$ which entails

$$E = \tfrac{n+2}{n}T = \tfrac{n+2}{2}U$$

For an oscillator we therefore have

$$E = 2 \cdot \frac{\ell^2}{2\mu r_0^2} = 2 \cdot \frac{\mu\omega^2 r_0^2}{2}$$

$$\Downarrow$$

$$r_0^2 = \ell/\mu\omega$$

$$\Downarrow$$

$$E = \ell\omega$$

[13] We will return also to this topic in §7, but in the meantime see Goldstein[12] or THERMODYNAMICS & STATISTICAL MECHANICS (2002), pages 162–164.

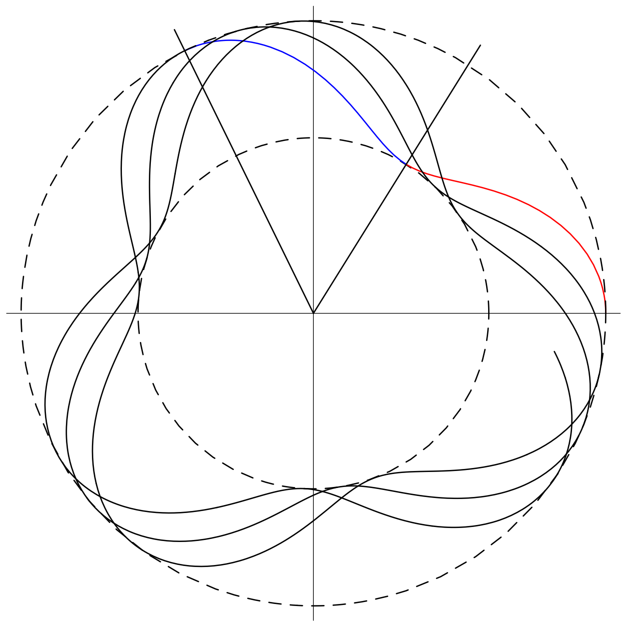

FIGURE 6: *Typical bound orbit, with θ advancing as r oscillates between $r_{\max}$ and $r_{\min}$, the total advance per oscillation being given by (25). In the figure, radials mark the halfway point and end of the first such oscillation.*

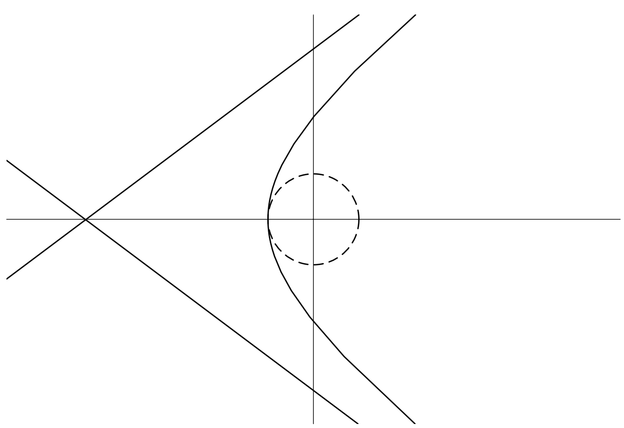

FIGURE 7: *Typical unbounded orbit, and its asymptotes. The angle between the asymptotes (scattering angle) can be computed from (26). The dashed circle (radius $r_{\min}$) marks the closest possible approach to the force center, which is, of course, $\{E, \ell\}$-dependent.*

which by the quantization rule $\ell \mapsto n\hbar$ would give

$$E = n\hbar\omega \quad : \quad n = 0, 1, 2, \ldots$$

Similarly, in case of the Kepler problem we have

$$\begin{aligned} E = -\frac{\ell^2}{2\mu r_0^2} &= \tfrac{1}{2}(-k/r_0) \\ &\Downarrow \\ r_0 &= \frac{\ell^2}{\mu k} \\ \Downarrow \qquad & \\ E = -\frac{\mu k^2}{2\ell^2} & \end{aligned}$$

which upon formal quantization yields Bohr's

$$E = -\frac{\mu k^2}{2\ell^2}\frac{1}{n^2} \quad : \quad n = 1, 2, 3, \ldots$$

If E is so large as to cause the orbit to be unbounded then (questions of closure and periodicity do not arise, and) an obvious modification of (25) supplies

$$\begin{aligned} \Delta\theta &\equiv \text{scattering angle} \\ &= 2\int_{r_{\min}}^{\infty} \frac{\ell/\mu r^2}{\sqrt{\frac{2}{\mu}\left[E - U(r) - \frac{\ell^2}{2\mu r^2}\right]}}\, dr \end{aligned} \tag{26}$$

Even for simple power-law potentials $U = kr^n$ the integrals (25) and (26) are analytically intractable except in a few cases (so say the books, and *Mathematica* appears to agree). Certainly analytical integration is certainly out of the question when $U(r)$ is taken to be one or another of the standard phenomenological potentials—such, for example, as the Yukawa potential

$$U(r) = -\frac{ke^{-\lambda r}}{r}$$

But in no concrete case does numerical integration pose a problem.

3. Orbital design. We learned already in Chapter 1 to distinguish the *design of a trajectory* (or orbit) from *motion along a trajectory*. We have entered now into a subject area which in fact sprang historically from a statement concerning orbital design: the 1st Law (1609) of Johannes Kepler (1571–1630) asserts that "planetary orbits are elliptical, with the sun at one focus." We look to what can be said about orbits more generally (other central force laws).

We had at bottom of page 13 a differential equation satisfied by $\theta(r)$. For many purposes it would, however, be more convenient to talk about $r(\theta)$, and in pursuit of that objective it would be very much to our benefit if we could find some way to avoid the functional inversion problem presented by $\theta(r)$. To that end we return to the Lagrangian (17), which supplies

$$\begin{aligned}\mu\ddot{r} - \mu r\dot{\theta}^2 &= -\tfrac{d}{dr}U(r)\\ &\equiv f(r)\end{aligned}$$

But at (20) we had $\dot{\theta} = \ell/\mu r^2$, so

$$\mu\ddot{r} - \ell^2/\mu r^3 = f(r)$$

From (20) it follows also that $\frac{d}{dt} = (\ell/\mu r^2)\frac{d}{d\theta}$ so we have

$$(\ell^2/\mu)\big[\tfrac{1}{r^2}\tfrac{d}{d\theta}\tfrac{1}{r^2}\tfrac{d}{d\theta}r - (1/r^3)\big] = f(r)$$

Introduce the new dependent variable $u = 1/r$ and obtain

$$\big[u^2\tfrac{d}{d\theta}u^2\tfrac{d}{d\theta}\tfrac{1}{u} - u^3\big] = (\mu/\ell^2)f(\tfrac{1}{u})$$

whence

$$\begin{aligned}\frac{d^2u}{d\theta^2} + u &= (\mu/\ell^2)\tfrac{1}{u^2}f(\tfrac{1}{u})\\ &= -(\mu/\ell^2)\frac{dU(\frac{1}{u})}{du}\end{aligned} \tag{27.1}$$

For potentials of the form $U(r) = kr^n$ we therefore have

$$= +n(k\mu/\ell^2)\,u^{-n-1} \tag{27.2}$$

The most favorable cases, from this point of view, are $n = -1$ and $n = -2$.

EXAMPLE: Harmonic central potential We look to the case $U(r) = \frac{1}{2}\mu\omega^2r^2$ where (27.2) reads

$$\frac{d^2u}{d\theta^2} + u = (\mu\omega/\ell)^2\,u^{-3} \tag{28}$$

and, because our objective is mainly to check the accuracy of recent assertions, we agree to "work backwards;" *i.e.*, from foreknowledge of the elementary fact that the orbit of an isotropic oscillator is a centered ellipse. In standard orientation

$$\begin{aligned}&\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\\ &\qquad\Downarrow\\ &r(\theta) = \sqrt{\frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}}\end{aligned}$$

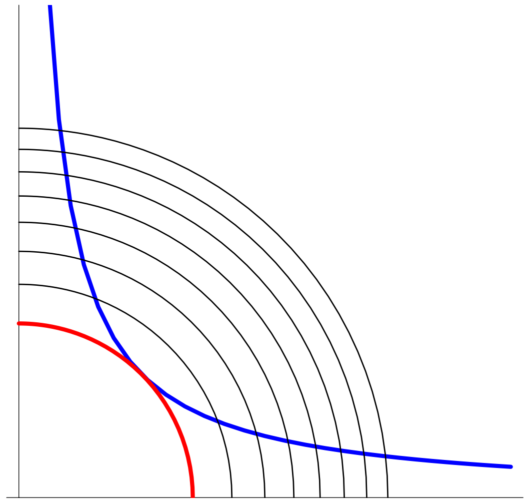

FIGURE 8: *Figure derived from* (29), *inscribed on the* $\{a, b\}$*-plane, shows a hyperbolic curve of constant angular momentum and several circular arcs of constant energy. The energy arc of least energy intersects the* ℓ*-curve at* $a = b$: *the associated orbit is circular.*

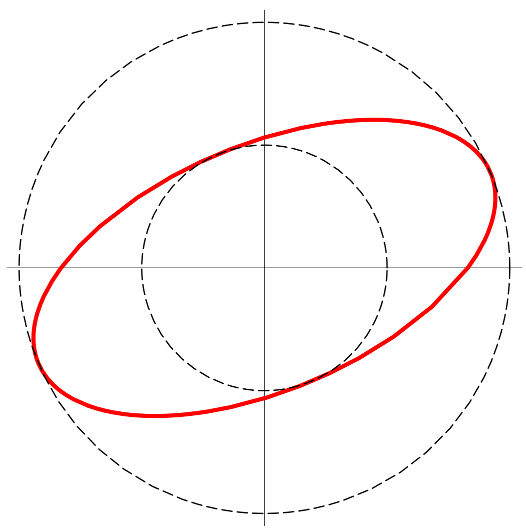

FIGURE 9: *Typical centered elliptical orbit of an isotropic harmonic oscillator, showing circles of radii* $r_{\max} = a$ *and* $r_{\min} = b$. *The isotropic oscillator is exceptional* (*though not quite unique*) *in that for this as* <u>*for all orbits*</u> *the angular advance per radial oscillation is* $\Delta\theta = \pi$: *all orbits close after a single circuit.*

so for an arbitrarily oriented centered ellipse we have

$$u(\theta) = \sqrt{\frac{b^2\cos^2(\theta-\delta) + a^2\sin^2(\theta-\delta)}{a^2b^2}}$$

Mathematica informs us that such functions satisfy (28) with

$$\ell = \mu\omega ab \tag{29.1}$$

Such an orbit is pursued with energy

$$E = \tfrac{1}{2}\mu\omega^2(a^2+b^2) \tag{29.2}$$

From (29) we obtain

$$\begin{aligned} a^2 &= \frac{E}{\mu\omega^2}\left[1 \pm \sqrt{1-\left(\frac{\ell\omega}{E}\right)^2}\,\right] \\ b^2 &= \frac{E}{\mu\omega^2}\left[1 \mp \sqrt{1-\left(\frac{\ell\omega}{E}\right)^2}\,\right] \end{aligned}$$

Evidently circular orbits ($a = b$) require that E and ℓ stand in the relation $E = \ell\omega$ encountered already at the bottom of page 15 (see FIGURE 8). Returning with (29) to (25) we find that the angular advance per radial oscillation is given by

$$\begin{aligned} \Delta\theta &= 2\int_b^a \frac{ab/r^2}{\sqrt{a^2+b^2-r^2-a^2b^2/r^2}}\,dr \\ &= \pi \quad : \quad \text{all } \{a,b\}\text{, by numerical experimentation} \end{aligned}$$

which simply reaffirms what we already knew: all isotropic oscillator orbits are closed/periodic (see FIGURE 9).

4. The Kepler problem: attractive $1/r^2$ force. Here

$$U(r) = -k\tfrac{1}{r} \quad : \quad k > 0$$

and the orbital equation (27.2) reads

$$\begin{aligned} \frac{d^2u}{d\theta^2} + u &= (k\mu/\ell^2)\,u^0 \\ &\equiv p \end{aligned}$$

or again

$$\frac{d^2v}{d\theta^2} + v = 0 \quad \text{with} \quad v \equiv u - p$$

Immediately $v(\theta) = q\cos(\theta-\delta)$ so

$$\begin{aligned} r(\theta) &= \frac{1}{p + q\cos(\theta-\delta)} \\ &= \frac{\alpha}{1+\varepsilon\cos(\theta-\delta)} \quad : \quad \text{more standard notation} \end{aligned} \tag{30}$$

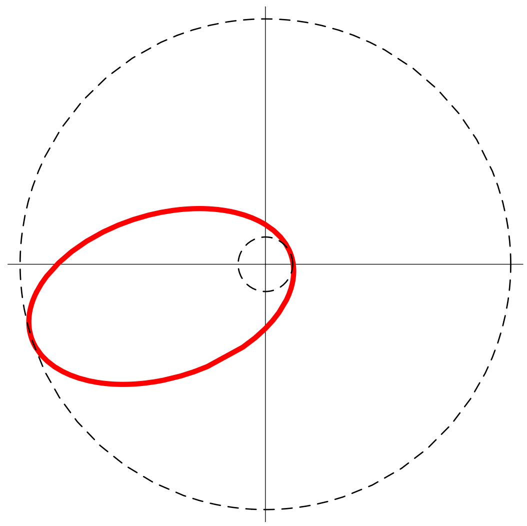

FIGURE 10: *Keplerian ellipse* (30) *with eccentricity* $\varepsilon = 0.8$. *The circles have radii*

$$r_{\text{min}} = \frac{\alpha}{1+\varepsilon} \quad : \quad \textit{“pericenter”}$$
$$r_{\text{max}} = \frac{\alpha}{1-\varepsilon} \quad : \quad \textit{“apocenter”}$$

When the sun sits at the central focus the “pericenter/apocenter” become the “perihelion/aphelion,” while if the earth sits at the focus one speaks of the “perigee/apogee.” It is clear from the figure, and an immediate implication of (30), that

$$\Delta\theta = 2\pi$$

Equation (30) provides—as Kepler asserted, as a little experimentation with *Mathematica*'s `PolarPlot[`*etc*`]` would strongly suggest, and as will presently emerge—the polar description of an ellipse of eccentricity ε with one focus at the force center (*i.e.*, at the origin).

To figure out how α and ε depend upon energy and angular momentum we return to (23) which gives

$$\begin{aligned}
\theta &= \int \frac{r^{-2}}{\sqrt{p + qr^{-1} - r^{-2}}}\,dr \\
&= -\int \frac{1}{\sqrt{p + qu - u^2}}\,du \\
&= \arctan\left\{\frac{q-2u}{2\sqrt{p+qu-u^2}}\right\} \\
&= \arcsin\left\{\frac{q-2u}{\sqrt{q^2+4p}}\right\}
\end{aligned}$$

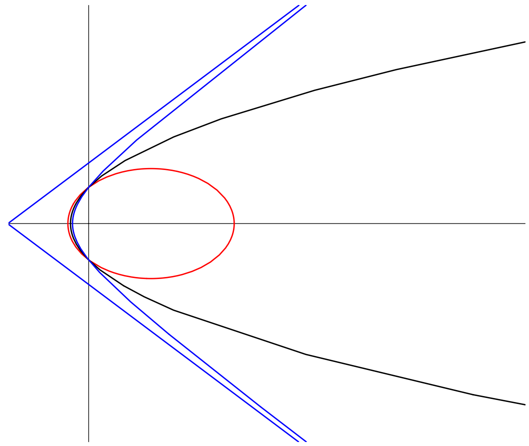

FIGURE 11: *Graphs of confocal Keplerian conic sections*

$$r = \frac{1}{1 - \varepsilon \cos \theta}$$

with $\varepsilon = 0.75, 1.00, 1.25$.

where now $p \equiv 2E\mu/\ell^2$ and $q \equiv 2k\mu/\ell^2$. So we have

$$u = \tfrac{1}{2}q - \tfrac{1}{2}\sqrt{q^2 + 4p}\,\sin\theta$$

and have only to adjust the point from which we measure angle $(\theta \mapsto \theta - \frac{1}{2}\pi)$ to recover (30) with (see the figure)

$$\begin{aligned} \alpha &= 2/q = \frac{\ell^2}{\mu k} \\ 0 \leqslant \varepsilon &= \sqrt{1 + 4p/q^2} = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}} \quad \begin{cases} < 1 & : \text{ ellipse} \\ = 1 & : \text{ parabola} \\ > 1 & : \text{ hyperbola} \end{cases} \end{aligned} \tag{31}$$

To achieve a circular orbit $(\varepsilon = 0)$ one must have

$$E = -\frac{\mu k^2}{2\ell^2} \quad i.e., \quad E\ell^2 = -\tfrac{1}{2}\mu k^2$$

which was encountered already on page 17, and which describes

- the least energy possible for a given angular momentum
- the greatest angular momentum possible for a given energy

for if these bounds were exceeded then ε would become imaginary.

The semi-major axis of the Keplerian ellipse is

$$\begin{aligned} a = \tfrac{1}{2}(r_{\min} + r_{\max}) &= \frac{\alpha}{1 - \varepsilon^2} \\ &= -\frac{k}{2E} \quad : \quad \text{positive because } E < 0 \end{aligned} \tag{32.1}$$

while the semi-minor axis (got by computing the maximal value assumed by $r(\theta)\sin\theta$) is

$$\begin{aligned} b &= \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \sqrt{a\alpha} \qquad (32.2)\\ &= \frac{\ell}{\sqrt{-2\mu E}} \quad : \quad \text{real for that same reason} \end{aligned}$$

The distance from center to focus is $f = \varepsilon a$ so the distance from focus to apocenter is $(1-\varepsilon)a = \alpha/(1+\varepsilon) = r_{\text{min}}$: this little argument serves to establish that the force center really does reside at a focal point.

Concerning *secular progress along an orbit*: the area swept out by r is

$$A(\theta) = \tfrac{1}{2}\int^{\theta} r^2(\vartheta)\,d\vartheta$$

so the rate of growth of A is

$$\dot{A} = \tfrac{1}{2}r^2\dot{\theta} = \tfrac{1}{2\mu}\ell \quad : \quad \text{constant for } \textit{every} \text{ force law} \qquad (33)$$

Multiplication by the period τ gives $\frac{1}{2\mu}\ell\tau = \pi ab = \pi a\sqrt{a\alpha}$ whence (by (31))

$$\tau^2 = \frac{4\pi^2\mu^2}{\ell^2}\alpha a^3 = \frac{4\pi^2\mu}{k}a^3 \qquad (34)$$

In the gravitational case one has

$$\frac{\mu}{k} = \frac{1}{Gm_1m_2}\cdot\frac{m_1m_2}{m_1+m_2} = \frac{1}{G(m_1+m_2)}$$

If one had in mind a system like the sun and its several lesser planets one might write

$$\Big(\frac{\tau_1}{\tau_2}\Big)^2 = \frac{M+m_2}{M+m_1}\Big(\frac{a_1}{a_2}\Big)^3$$

and with the neglect of the planetary masses obtain Kepler's 3rd Law (1619)

$$\Big(\frac{\tau_1}{\tau_2}\Big)^2 \approx \Big(\frac{a_1}{a_2}\Big)^3 \qquad (35)$$

It is interesting to note that the harmonic force law would have supplied, by the same reasoning (but use (29.1)), $\frac{1}{2\mu}\ell\tau = \pi ab = \pi\ell/\mu\omega$ whence

$$\tau = 2\pi/\omega \quad : \quad \text{all values of } E \text{ and } \ell$$

We have now in hand, as derived consequences of Newton's Laws of Motion and Universal Law of Gravitation,

- KEPLER'S FIRST LAW OF PLANETARY MOTION: Planets pursue elliptical orbits, with the sun at one focus;
- KEPLER'S SECOND LAW: The radius sweeps out equal areas in equal times;
- KEPLER'S THIRD LAW: For any pair of planets, the square of the ratio of periods equals the cube of the ratio of semi-major axes.

Kepler's accomplishment is really quite amazing. He worked with data—relating mainly to the orbit of Mars—inherited from Tycho Brahe (1546–1601), who was a naked-eye astronomer, working before the invention of the telescope, and even without the benefit of reliable clocks. Logarithms were at the time a new invention, and Kepler had to construct his own log tables. The ellipticity of Mars' orbit (which is strongly perturbed by Jupiter) is relatively pronounced ($\varepsilon_{\text{Mars}}$ ranges between 0 and 0.14, while $\varepsilon_{\text{Earth}}$ ranges between 0 and 0.06), yet it was radical to suggest that planetary orbits were elliptical, when it had been accepted on good authority for many centuries that they were epicyclic—assembled from perfect circles, as befits the perfect heavens. Kepler worked *from data*, without the support of a theoretical dynamics—that development had to wait seventy-five years for Newton to complete his work. Newton cites Kepler's accomplishment as a principal motivation in the opening pages of the *Principia*, and considered his ability to account theoretically for Kepler's laws to be persuasive evidence of his own success: when he remarked that it had been his good fortune to "stand on the shoulders of giants" it was Copernicus, Galileo and (I suspect especially) Kepler that he had in mind. But Kepler himself died ignorant of (among other implications of his work) the facts that—while his 1$^{\text{st}}$ and 3$^{\text{rd}}$ Laws are specific to $1/r^2$ attractive interactions—his 2$^{\text{nd}}$ Law is a statement simply of the conservation of angular momentum, and holds for *all* central forces. So, for that matter, did Newton: the concept of "angular momentum" was not invented until about twenty-five years after Newton's death. By Euler.

5. Kepler's equation. Planetary astronomers used to—and perhaps still do—have practical reason to construct FIGURE 12. Introducing an angle

$$\tau \equiv 2\pi\frac{t}{\text{period}} \quad : \quad \text{clock started at pericenter}$$

(known to astronomers as the "mean anomaly") and importing from the physics of the situation only Kepler's 2$^{\text{nd}}$ Law, one arrives at "Kepler's equation" (also called "the equation of time")

$$\tau = \theta_0 - \varepsilon \sin\theta_0 \tag{36}$$

The problem—first confronted by Kepler—is to achieve the functional inversion of (36), for if that could be accomplished then one could insert $\theta_0(\tau)$ onto (37) to obtain a polar description of the *motion of the planet along its elliptial orbit*.

I have read that more than 600 solutions of—we would like to call it "Kepler's problem"—have been published in the past nearly 400 years, many of them by quite eminent mathematicians (Lagrange, Gauss, Cauchy, Euler, Levi-Civita). Those have been classified and discussed in critical detail in a

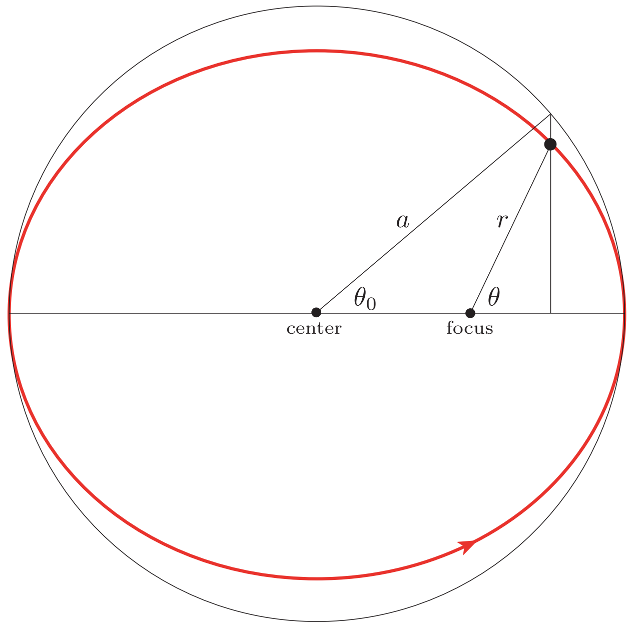

FIGURE 12: *Circle inscribed about a Keplerian ellipse, showing the relation of the "eccentric anomaly" θ_0 to the "true anomaly" θ. One can show, using only elementary relations standard to the geometrical theory of ellipses, that*

$$\left.\begin{aligned} r &= a(1-\varepsilon\cos\theta_0)\\ \tan\tfrac{1}{2}\theta &= \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\tfrac{1}{2}\theta_0 \end{aligned}\right\} \tag{37}$$

These equations serve, in effect, to provide a parametric description $\{r(\theta_0),\theta(\theta_0)\}$ *of the polar representation of a Keplerian ellipse (by which phrase I mean an ellipse with one focus at the polar origin). Elimination of the parameter would give back*

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}$$

which by (32.1) *is equivlent to* (30): *case* $\delta = 0$.

recent quite wonderful book.[14] I propose to sketch Kepler's own solution and the approach to the problem that led Bessel to the invention of Bessel functions.

---

[14] Peter Colwell, *Solving Kepler's Equation over Three Centuries* (1993). Details of the arguments that lead to (36) and (36) can be found there; also in the "Appendix: Historical introduction to Bessel functions" in RELATIVISTIC CLASSICAL FIELD THEORY (1973), which provides many references to the astronomical literature.

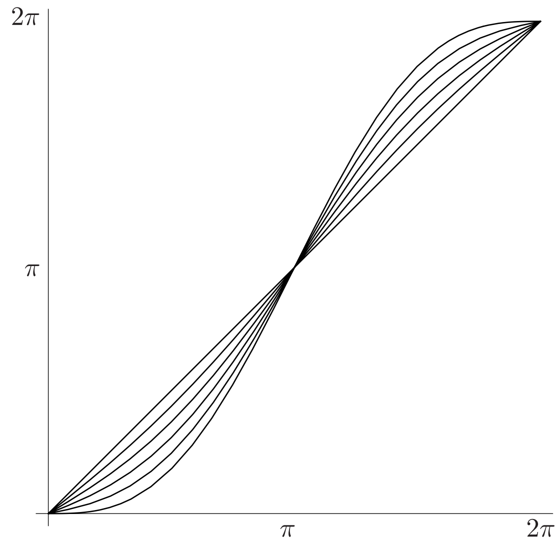

FIGURE 13: *Kepler's function* $K(x;\varepsilon) \equiv x - \varepsilon \sin x$, *shown with* $\varepsilon = 0, 0.2, 0.4, 0.6, 0.8, 1.0$.

To solve $y = K(x;\varepsilon)$ Kepler would first guess a solution (the "seed" x_0) and then compute

$$\begin{aligned} y_0 &= K(x_0) \\ y_1 &= K(x_1) \quad \text{with} \quad x_1 = x_0 + (y - y_0) \\ y_2 &= K(x_2) \quad \text{with} \quad x_2 = x_1 + (y - y_1) \\ &\vdots \end{aligned}$$

EXAMPLE: Suppose the problem is to solve $1.5000 = K(x;0.2)$. To the command

```
FindRoot[1.5000==K[x,0.2], {x,1.5}]
```

Mathematic responds $\quad$ `x → 1.69837`

Kepler, on the other hand—if he took $x_0 = 1.5000$ as his seed—would respond

$$\begin{aligned} 1.3005 &= K(1.5000) \\ &\qquad 1.5000 + (1.5000 - 1.3005) = 1.6995 \\ 1.5012 &= K(1.6995) \\ &\qquad 1.6995 + (1.5000 - 1.5012) = 1.6983 \\ 1.5000 &= K(1.6983) \end{aligned}$$

and get 4-place accuracy after only two iterations—even though $\varepsilon = 0.2$ is large by planetary standards. Colwell[14] remarks that both Kepler's equation and his method for solving it can be found in 9$^{\text{th}}$ Century work of one Habash-al-Hasib, who, however, took his motivation not from astronomy but from "problems of parallax."

We make our way now through the crowd at this convention of problem-solvers to engage Bessel[15] in conversation. Bessel's idea—which[16] iin retrospect seems so straightforward and natural—was to write

$$\theta_0(\tau)-\tau=\varepsilon\sin\theta_0(\tau)=2\sum_1^\infty B_n\sin n\tau$$

From the theory of Fourier series (which was a relative novelty in 1817) one has

$$\begin{aligned}
B_n &= \tfrac{1}{\pi}\int_0^\pi[\theta_0(\tau)-\tau]\sin n\tau\,d\tau\\
&= -\tfrac{1}{n\pi}\int_0^\pi[\theta_0(\tau)-\tau]\,d(\cos n\tau)\\
&= \underbrace{-\tfrac{1}{n\pi}[\theta_0(\tau)-\tau]\cos n\tau\Big|_0^\pi}_{=\,0\text{ because }\theta_0(\tau)-\tau\text{ vanishes at the apses}}+\tfrac{1}{n\pi}\int_0^\pi\cos n\tau\,d[\theta_0(\tau)-\tau]\\
&= \tfrac{1}{n\pi}\int_0^\pi\cos n\tau\,d\theta_0(\tau)-\tfrac{1}{n\pi}\underbrace{\int_0^\pi\cos n\tau\,d\tau}_{=\,0\text{ for }n=1,2,3\ldots}\\
&= \tfrac{1}{n\pi}\int_0^\pi\cos n(\theta_0-\varepsilon\sin\theta_0)\,d\theta_0
\end{aligned}$$

---

[15] Regarding the life and work of Friedrich Wilhelm Bessel (1784–1864): it was to prepare himself for work as a cabin-boy that, as a young man, he took up the study of navigation and practical astronomy. To test his understanding he reduced some old data pertaining to the orbit of Halley's comet, and made such a favorable impression on the astronomers of the day (among them Olbers) that in 1810, at the age of 26, he was named Director of the new Königsberg Observatory. Bessel was, therefore, a contemporary and respected colleague of K. F. Gauss (1777–1855), who was Director of the Göttingen Observatory. Bessel specialized in the precise measurement of stellar coordinates and in the observatio of binary stars: in 1838 he computed the distance of 61 Cygni, in 1841 he discovered the dark companion of Sirius, and in 1842 he determined the mass, volume and density of Jupiter. He was deeply involved also in the activity which led to the discovery of Neptune (1846). It was at about that time that Bessel accompanied his young friend Jacobi (1804–1851) to a meeting of the British Association—a meeting attended also by William Rowan Hamilton (1805–1865). Hamilton had at twenty-two (while still an undergraduate) been appointed Royal Astronomer of Ireland, Director of the Dunsink Observatory and Professor of Astronomy. His name will forever be linked with that of Jacobi, but on the occasion—the only time when Hamilton and Jacobi had an opportunity to exchange words face to face—Hamilton reportedly ignored Jacobi, and seemed much more interested in talking to Bessel.

[16] Note that $\theta_0-\tau$ is, by (36), an odd periodic function of θ_0, and therefore of τ.

—the variable-of-integration role having beentaken over here by θ_0, which ranges from 0 to π as τ does. Thus does Bessel obtain

$$B_n = \tfrac{1}{n} J_n(n\varepsilon)$$

where

$$J_n(x) \equiv \tfrac{1}{\pi}\int_0^\pi \cos(n\varphi - x\sin\varphi)\,d\varphi$$

serves to *define* the **Bessel function of integral order n**. Bessel's inversion of the Kepler equation can now be described

$$\theta_0(\tau) = \tau + 2\sum_1^\infty \tfrac{1}{n} J_n(n\varepsilon)\sin n\tau \tag{38}$$

If confronted with our **EXAMPLE** (page 26, Bessel would write[17]

$$\begin{aligned}
&\theta_0(1.50000\\
&\quad = 1.50000 + 2\Big\{0.09925 + 0.00139 - 0.00143 - 0.00007 + 0.00005 + \cdots\Big\}\\
&\quad = 1.69837
\end{aligned}$$

which is precise to all the indicated decimals. The beauty of (38) is, however, that it speaks simultaneously about the $\theta_0(\tau)$ that results from *every* value of τ, whereas Kepler's method requires one to reiterate at each new τ-value. For small values of ε Bessel's (38) supplies

$$\begin{aligned}
\theta_0(\tau) = \tau &+ \big\{\varepsilon - \tfrac{1}{8}\varepsilon^3 + \tfrac{1}{192}\varepsilon^5 + \ldots\big\}\sin\tau\\
&+ \big\{\tfrac{1}{2}\varepsilon^2 - \tfrac{1}{6}\varepsilon^4 + \cdots\big\}\sin 2\tau\\
&+ \big\{\tfrac{3}{8}\varepsilon^3 - \tfrac{27}{128}\varepsilon^5\big\}\sin 3\tau\\
&+ \big\{\tfrac{1}{3}\varepsilon^4 + \cdots\big\}\sin 4\tau + \cdots
\end{aligned}$$

Bessel pioneered the application of Fourier analysis to a *variety* of astronomical problems, and had more to say also about its application to the inversion of Kepler's equation: for discussion of the fascinating details, see pages 27–40 in Colwell.[14]

6. The Runge-Lenz vector. While the history of linear algebra is a famously tangled tale to which dozens of prominent mathematicians contributed (often contentiously), the history of what we have come to call "vector analysis" is a story of stark simplicity. The subject is the creation (during the 1880s) of one man—Josiah Willard Gibbs (1839–1903), whose Yankee intent was to extract from that jumbled wisdom a simple tool with the sharp practical utility of a scythe. The first public account of his work appeared in *Vector Analysis* by

[17] Use the *Mathematica* command `BesselJ[n,x]` to evaluate $J_n(x)$.

J. W. Gibbs & E. B. Wilson (1901),[18] and it is from §61 Example 3 that I take the following argument.[19]

Let a mass point μ move subject to the central force $\boldsymbol{F} = -\frac{k}{r^2}\hat{\boldsymbol{x}}$. From the equation of motion

$$\mu\ddot{\boldsymbol{x}} = -\tfrac{k}{r^3}\boldsymbol{x}$$

it follows that $\frac{d}{dt}(\boldsymbol{x}\times\mu\dot{\boldsymbol{x}}) = \boldsymbol{0}$, from which Gibbs obtains the angular momentum vector as a "constant of integration:"

$$\boldsymbol{x}\times\mu\dot{\boldsymbol{x}} = \boldsymbol{L} \quad : \quad \text{constant}$$

Gibbs (in a characteristic series of masterstrokes) invites us now to construct

$$\ddot{\boldsymbol{x}}\times\boldsymbol{L} = -\tfrac{k}{r^3}\,\boldsymbol{x}\times\boldsymbol{L}$$

and to notice that

$$\text{expression on the left} = \tfrac{d}{dt}\big\{\mu\dot{\boldsymbol{x}}\times\boldsymbol{L}\big\}$$

$$\begin{aligned}\text{expression on the right} &= -\tfrac{\mu k}{r^3}\boldsymbol{x}\times(\boldsymbol{x}\times\dot{\boldsymbol{x}})\\ &= -\tfrac{\mu k}{r^3}\big\{(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})\boldsymbol{x} - (\boldsymbol{x}\cdot\boldsymbol{x})\dot{\boldsymbol{x}}\big\}\\ &= -\tfrac{\mu k}{r^3}\big\{(r\dot{r})\boldsymbol{x} - r^2\dot{\boldsymbol{x}}\big\}\\ &= \tfrac{d}{dt}\big\{\mu k\tfrac{1}{r}\boldsymbol{x}\big\}\end{aligned}$$

entail

$$\dot{\boldsymbol{x}}\times\boldsymbol{L} = k\tfrac{1}{r}\boldsymbol{x} + \boldsymbol{K}$$

where

$$\boldsymbol{K} = \dot{\boldsymbol{x}}\times\boldsymbol{L} - k\tfrac{1}{r}\boldsymbol{x} \quad : \quad \text{constant of integration}$$

precisely reproduces the definition of a <u>constant of Keplerian motion additional to energy and angular momentum</u> that has become known as the "Runge-Lenz vector"... though as I understand the situation it was upon Gibbs that Runge patterned his (similarly pedagogical) discussion, and from Runge that Lenz borrowed the $\boldsymbol{K}$ that he introduced into the "old quantum theory of the

[18] The book—based upon class notes that Gibbs had developed over a period of nearly two decades—was actually written by Wilson, a student of Gibbs who went on to become chairman of the Physics Department at MIT and later acquired great distinction as a professor at Harvard. Gibbs admitted that he had not had time even to puruse the work before sending it to the printer. The book contains no bibliography, no reference to the literature apart from an allusion to work of Heaviside and Föpple which can be found in Wilson's General Preface.

[19] The argument was intended to illustrate the main point of §61, which is that "the ... *integration* of vector equations in which the differentials depend upon scalar variables needs but a word."

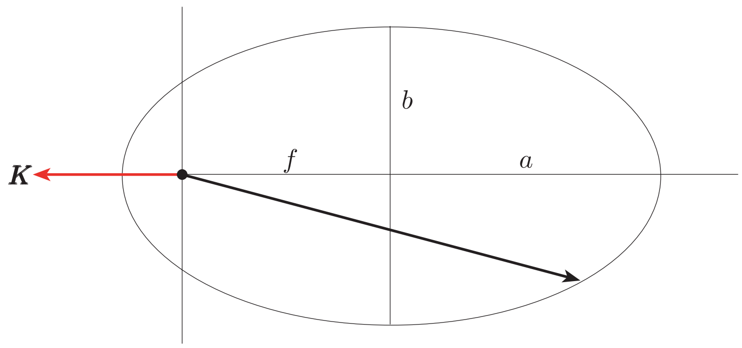

FIGURE 14: *Orientation of the Runge-Lenz vector in relation to the Keplerian ellipse along which the particle is moving.*

hydrogen atom," with results that are remembered only because they engaged the imagination of the young Pauli.[20]

What is the conservation of

$$\boldsymbol{K} = \tfrac{1}{\mu}\,\boldsymbol{p}\times\boldsymbol{L} - \tfrac{k}{r}\,\boldsymbol{x} \tag{39}$$

trying to tell us? Go to either of the apses (points where the orbit intersects the principal axis) and it becomes clear that

$\boldsymbol{K}$ runs parallel to the principal axis

(because at those points $\boldsymbol{p}\times\boldsymbol{L}$ and $\boldsymbol{x}$ both do). Dotting (39) into itself we get

$$\begin{aligned}\boldsymbol{K}\cdot\boldsymbol{K} &= \tfrac{1}{\mu^2}(\boldsymbol{p}\times\boldsymbol{L})\cdot(\boldsymbol{p}\times\boldsymbol{L}) - 2\tfrac{k}{\mu r}(\boldsymbol{p}\times\boldsymbol{L})\cdot\boldsymbol{x} + k^2\tfrac{1}{r^2}\boldsymbol{x}\cdot\boldsymbol{x}\\ &= \tfrac{1}{\mu^2}p^2\ell^2 - 2\tfrac{k}{\mu r}\ell p r + k^2 \quad \text{by evaluation at either of the apses}\\ &= \tfrac{2}{\mu}\big[\tfrac{1}{2\mu}p^2 - \tfrac{k}{r}\big]\ell^2 + k^2\end{aligned}$$

giving

$$K^2 = \tfrac{2}{m}E\ell^2 + k^2$$

which by (31) becomes

$$= (k\varepsilon)^2 \tag{40}$$

K is "uninteresting" in that its conserved value is implicit already in the conserved values of E and ℓ, but interestingly it involves those parameters

[20] In "Prehistory of the 'Runge-Lenz' vector" (AJP **43**, 737 (1975)) Goldstein traces the history of what he calls the "Laplace-Runge-Lenz vector" back to Laplace (1799). Reader response to that paper permitted him in a subsequent paper ("More on the prehistory of the Laplace-Runge-Lenz vector," AJP **44**, 1123 (1976)) to trace the idea back even further, to the work of one Jacob Hermann (1710) and its elaboration by Johann Bernoulli (1712).

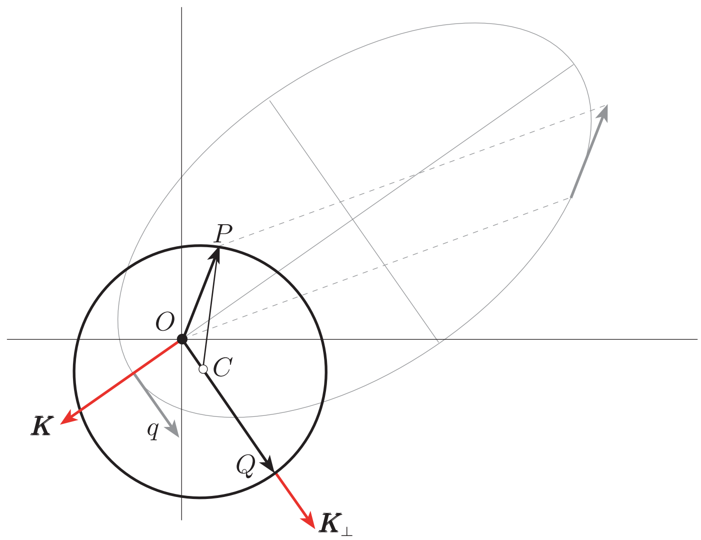

FIGURE 15: *Keplerian orbit $\mathcal{G}$ superimposed upon the hodograph $\mathcal{H}$. It was Hamilton's discovery that the Keplerian hodograph is circular, centered on a line which stands normal to the principal axis at the force center O. Q identifies the momentum at the pericenter, and q the associated orbital tangent. The dogleg construction*

$$OP = OC + CP$$

illustrates the meaning of (41)*, and the dashed lines indicate how points on the hodograph are to be associated with tangents to the orbit.*

only as they combine to describe the *eccentricity* of the Keplerian orbit. Additional light is cast upon the role of $\boldsymbol{K}$ by the following observations:

The motion of a particle in a central force field traces in its effectively 4-dimensional phase space a curve $\mathcal{C}$. Projection of $\mathcal{C}$ onto the $\boldsymbol{x}$-plane produces a curve $\mathcal{G}$ familiar as the "trajectory" (or "orbit") of the particle. Projection onto the $\boldsymbol{p}$-plane produces a less familiar curve $\mathcal{H}$ called the "hodograph." In the case of central $1/r^2$ forces the curves $\mathcal{G}$ are of course just the ellipses/parabolas/hyperbolas identified by Kepler and reproduced by Newton, but the associated "Keplerian hodographs" were apparently first studied by Hamilton[21] (who gave such curves their name). Working from (39), we have

$$\boldsymbol{K}_\perp \equiv \boldsymbol{L} \times \boldsymbol{K} = \tfrac{1}{\mu}\ell^2 \boldsymbol{p} - \tfrac{k}{r}\boldsymbol{L} \times \boldsymbol{x}$$

[21] See Chapter 24 of T. L. Hankins' *Sir William Rowan Hamilton* (1980) and the second of the Goldstein papers previously cited.[20] It was in connection with this work—inspired by the discovery of Neptune (1846)—that Hamilton was led to the independent (re/pre)invention of the "Hermann-. . . -Lenz" vector.

giving

$$\begin{aligned}\boldsymbol{p} &= (\mu/\ell^2)\boldsymbol{K}_\perp + (\mu k/\ell^2)\boldsymbol{L}\times\hat{\boldsymbol{x}}\\ &= (\text{constant vector of length } \tfrac{\mu K}{\ell})\\ &\qquad + (\text{ vector that traces a circle of radius } \tfrac{\mu k}{\ell})\end{aligned}$$

From (40) it now follows that

$$(\text{radius})^2 - (\text{displacement})^2 = -2\mu E \quad \begin{cases} > 0 & : \quad \text{elliptical orbit}\\ = 0 & : \quad \text{parabolic orbit}\\ < 0 & : \quad \text{hyperbolic orbit}\end{cases}$$

We are brought thus to the striking conclusion—illustrated in FIGURE 15, and apparently overlooked by Newton—that the Keplerean hodograph $\mathcal{H}$ is in all cases *circular*, and envelops the origin (or doesn't) according as the trajectory $\mathcal{G}$ is bound (or unbound).

Bringing (40) to (39) we have

$$k\varepsilon\hat{\boldsymbol{K}} = \tfrac{1}{\mu}\,\boldsymbol{p}\times\boldsymbol{L} - \tfrac{k}{r}\,\boldsymbol{x}$$

which when dotted into $\boldsymbol{x}$ gives

$$\begin{aligned}k\varepsilon r\cos\theta &= \tfrac{1}{\mu}\boldsymbol{x}\cdot(\boldsymbol{p}\times\boldsymbol{L}) - kr\\ &= \tfrac{1}{\mu}\boldsymbol{L}\cdot(\boldsymbol{x}\times\boldsymbol{p}) - kr\\ &= \tfrac{1}{\mu}\ell^2 - kr\\ &\Downarrow\\ r &= \frac{\ell^2/\mu k}{1+\varepsilon\cos\theta}\end{aligned} \tag{41}$$

And this, we have known since (30/31), provides a polar description of the Keplerian *orbit*. It is remarkable that $\boldsymbol{K}$ provides a magical high-road to the construction of *both* $\mathcal{G}$ and $\mathcal{H}$.

We are in position now to clear up a dangling detail: at the pericenter (39) assumes the form

$$\begin{aligned}\boldsymbol{K} &= (\text{vector of length } \ell^2/\mu r_{\text{min}} \text{ directed toward the pericenter})\\ &\qquad + (\text{vector of length } k \text{ directed toward the apocenter})\end{aligned}$$

But from (41) it follows that

$$\ell^2/\mu r_{\text{min}} = k(1+\varepsilon) \geqslant k$$

so for non-circular orbits ($\varepsilon > 0$) the former vector predominates: the $\boldsymbol{K}$ vector is directed <u>toward the pericenter</u>, as was indicated in FIGURE 14. From results now in hand it becomes possible to state that the *center* of the orbital ellipse resides at

$$\boldsymbol{C} = -f\hat{\boldsymbol{K}} = -(f/k\varepsilon)\boldsymbol{K} = -(a/k)\boldsymbol{K}$$

It will be appreciated that the invariance of the Runge-Lenz vector is a property special to the Kepler problem. Quite generally,

$$\dot{\boldsymbol{K}} = [\boldsymbol{K}, H\,]$$

and if[22]

$$H = \tfrac{1}{2\mu}\boldsymbol{p}\cdot\boldsymbol{p} - \kappa(\boldsymbol{x}\cdot\boldsymbol{x})^{-n/2}$$

then (with major assistance by *Mathematica*) we compute

$$\begin{aligned}\dot{\boldsymbol{K}} &= \frac{kr^n - n\kappa r}{\mu\, r^{3+n}}\;(\boldsymbol{x}\times\boldsymbol{L})\\ &= 0 \quad \text{if and only if } n = 1 \text{ and } k = \kappa\end{aligned}$$

It seems natural to suppose that by watching the motion of $\boldsymbol{K}$ we could get a natural handle on the *orbital precession* that results (except in the harmonic case $n = -4,\ \kappa = -\mu\omega^2$) when $n \neq 1$. I mention in this connection that if were to define

$$\boldsymbol{\mathcal{K}}_n \equiv \tfrac{1}{\mu}\,\boldsymbol{p}\times\boldsymbol{L} - (\kappa/r^n)\,\boldsymbol{x}$$

then, by computation, we would have

$$\dot{\boldsymbol{\mathcal{K}}}_n = [\,\boldsymbol{\mathcal{K}}_n, H\,] = \frac{(n-1)\,\kappa}{\mu}\,\frac{\boldsymbol{p}}{r^n}$$

—the implication being that $\boldsymbol{\mathcal{K}}_n$ may be a more natural object to discuss in such a connection than $\boldsymbol{K}$ itself.

6. Accidental symmetry. Conservation laws speak to us of *symmetry*. When a particle m moves in the presence of an impressed central force $\boldsymbol{F} = -\boldsymbol{\nabla}U(r)$ we expect generally to have four conservation laws

$$[H, H] = 0 \quad \text{and} \quad [H, \boldsymbol{L}\,] = \mathbf{0}$$

but in the Keplerian case

$$H = \tfrac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} - \frac{k}{\sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}}$$

we have an additional three:

$$[H, \boldsymbol{K}\,] = \mathbf{0} \tag{42}$$

[22] Notice that $[\kappa/r^n] = [k/r] =$ energy.

To what symmetry can (42) refer? That $\boldsymbol{L}$-conservation refers to the rotational symmetry of the system can be construed to follow from the observation that the Poisson bracket algebra

$$\begin{aligned}
[L_1, L_2] &= L_3 \\
[L_2, L_3] &= L_1 \\
[L_3, L_1] &= L_2
\end{aligned}$$

is identical to the commutator algebra satisfied by the antisymmetric generators of 3×3 rotation matrices: write

$$\mathbb{R} = e^{\mathbb{A}} \quad \text{with} \quad \mathbb{A} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} = a_1\mathbb{L}_1 + a_2\mathbb{L}_2 + a_3\mathbb{L}_3$$

and observe that

$$\begin{aligned}
[\,\mathbb{L}_1, \mathbb{L}_2] &= \mathbb{L}_3 \\
[\,\mathbb{L}_2, \mathbb{L}_3] &= \mathbb{L}_1 \\
[\,\mathbb{L}_3, \mathbb{L}_1] &= \mathbb{L}_2
\end{aligned}$$

Thus inspired, we compute[23]

$$[L_1, K_1] = [L_2, K_2] = [L_3, K_3] = 0$$

$$\begin{aligned}
[L_1, K_2] &= +K_3 \qquad & [L_1, K_3] &= -K_2 \\
[L_2, K_3] &= +K_1 \qquad & [L_2, K_1] &= -K_3 \\
[L_3, K_1] &= +K_2 \qquad & [L_3, K_2] &= -K_1
\end{aligned}$$

$$\begin{aligned}
[K_1, K_2] &= (-2H/m)L_3 \\
[K_2, K_3] &= (-2H/m)L_1 \\
[K_3, K_1] &= (-2H/m)L_2
\end{aligned}$$

Defining

$$\boldsymbol{J} \equiv \boldsymbol{K}/\sqrt{-2H/m} \tag{43}$$

we therefore have

$$\left.\begin{aligned}
[L_i, L_j] &= \epsilon_{ijk}L_k \\
[L_i, J_j] &= \epsilon_{ijk}J_k \\
[\,J_i, J_j] &= \epsilon_{ijk}L_k
\end{aligned}\right\} \tag{44}$$

From (44) it quickly follows, by the way, that

$$\begin{aligned}
[L^2, \boldsymbol{L}\,] &= \boldsymbol{0} \\
[L^2, \boldsymbol{J}\,] &= -2\,\boldsymbol{L}\times\boldsymbol{J} \\
[J^2, \boldsymbol{L}\,] &= \boldsymbol{0} \\
[J^2, \boldsymbol{J}\,] &= +2\,\boldsymbol{L}\times\boldsymbol{J}
\end{aligned}$$

[23] The calculation is enormously tedious if attempted by hand, but presents no problem at all to *Mathematica*.

and that

$$[L^2, J^2] = 0$$

with $L^2 \equiv \boldsymbol{L}\cdot\boldsymbol{L}$ and $J^2 \equiv \boldsymbol{J}\cdot\boldsymbol{J}$, but heavy calculation is required to establish finally that

$$L^2 + J^2 = -k^2\big[\tfrac{2}{m}H\big]^{-1} \tag{45}$$

On a hunch we now write

$$\begin{pmatrix} 0 & -b_1 & -b_2 & -b_3 \\ b_1 & 0 & -a_3 & +a_2 \\ b_2 & +a_3 & 0 & -a_1 \\ b_3 & -a_2 & +a_1 & 0 \end{pmatrix} = a_1\mathbb{L}_1 + a_2\mathbb{L}_2 + a_3\mathbb{L}_3 + b_1\mathbb{J}_1 + b_2\mathbb{J}_2 + b_3\mathbb{J}_3$$

and observe that the computed commutation relations

$$\left.\begin{aligned} [\mathbb{L}_i, \mathbb{L}_j] &= \epsilon_{ijk}\mathbb{L}_k \\ [\mathbb{L}_i, \mathbb{J}_j] &= \epsilon_{ijk}\mathbb{J}_k \\ [\mathbb{J}_i, \mathbb{J}_j] &= \epsilon_{ijk}\mathbb{L}_k \end{aligned}\right\} \tag{46}$$

are structurally identical to (44). The clear implication is that the "accidental" constants of motion $J_1(\boldsymbol{x},\boldsymbol{p})$, $J_2(\boldsymbol{x},\boldsymbol{p})$, $J_3(\boldsymbol{x},\boldsymbol{p})$ have joined $L_1(\boldsymbol{x},\boldsymbol{p})$, $L_2(\boldsymbol{x},\boldsymbol{p})$, $L_3(\boldsymbol{x},\boldsymbol{p})$ to lead us beyond the group $O(3)$ of spherical symmetries written onto the face of *every* central force system. . . to a group $O(4)$ of canonical transformations that live in the 6-dimensional phase space of the system. The $L_i(\boldsymbol{x},\boldsymbol{p})$ fit naturally within the framework provided by Noether's theorem, but the generators $J_i(\boldsymbol{x},\boldsymbol{p})$ refer to a symmetry that lies beyond Noether's reach.

The situation is clarified if one thinks of all the Keplerian ellipses that can be inscribed on some given/fixed orbital plane, which we can without loss of generality take to be the $\{x_1, x_2\}$-plane. The lively generators are then

$$L_3(x_1,x_2,p_1,p_2),\ \ J_1(x_1,x_2,p_1,p_2) \text{ and } J_2(x_1,x_2,p_1,p_2),$$

which support the closed Poisson bracket sub-algebra

$$\begin{aligned} [J_1, J_2] &= L_3 \\ [J_2, L_3] &= J_1 \\ [L_3, J_1] &= J_2 \end{aligned}$$

To emphasize the evident fact that we have now in hand the generators of another copy of $O(3)$ we adjust our notation

$$\begin{aligned} J_1 &\to S_1 \\ J_2 &\to S_2 \\ L_3 &\to S_3 \end{aligned}$$

so that the preceding relations become simply

$$[S_i, S_j] = \epsilon_{ijk}S_k \tag{47.1}$$

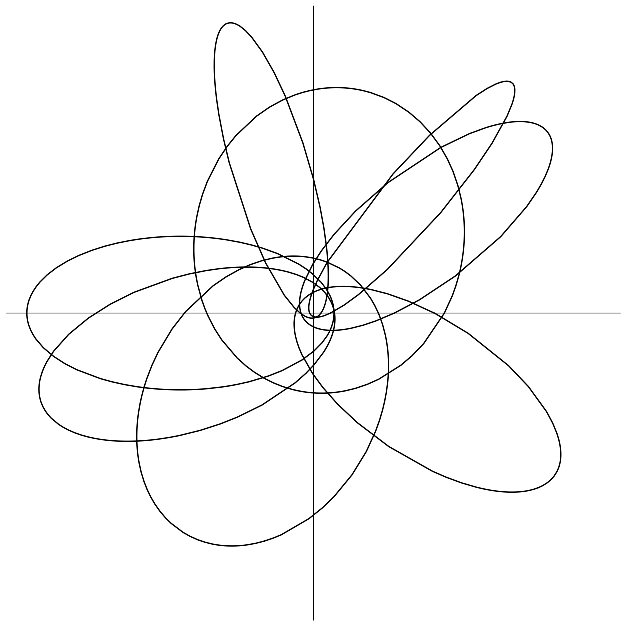

FIGURE 16: *Confocal population $\{\mathcal{G}\}_E$ of isoenergetic Keplerian orbits. The population is mapped onto itself under action of the generating observables $\{S_1, S_2, S_3\}$. The "isoenergetic" presumption is reflected in the circumstance that all such ellipses have the same semi-major axis.*

while (45) becomes

$$S_1^2 + S_2^2 + S_3^2 = S_0^2 \quad \text{with} \quad S_0^2 \equiv -k^2 m/2H \tag{47.2}$$

We have arrived here at apparatus very similar to that which Stokes/Poincaré devised to describe the states of elliptically polarized light. The observables $\{S_1, S_2, S_3\}$—of which, by (47.2), only two are, at given energy, independent—generate canonical transformations that serve to map onto itself the set $\{\mathcal{C}\}_E$ of all Keplerian curves inscribed within a certain 4-dimensional subspace of 6-dimensional phase space. Projected onto the $\{x_1, x_2\}$-plane, $\{\mathcal{C}\}_E$ becomes the set $\{\mathcal{G}\}_E$ of all isoenergetic Keplerian orbits (FIGURE 16), and when projected onto the $\{p_1, p_2\}$-plane it becomes the companion set $\{\mathcal{H}\}_E$ of all Keplerian hodographs.

One major detail remains to be discussed (plus any number of smaller ones with which I will not test my reader's patience). We have tacitly restricted our attention thus far to *closed* Keplerian orbits (it being the atypical/accidental closure of such orbits that makes the whole exercise possible!). For closed orbits $E < 0$, so the observables $\boldsymbol{J}$ introduced at (43) are real. But for hyperbolic orbits $E > 0$ and we are forced to adjust the definition, writing

$$\boldsymbol{J} \equiv \boldsymbol{K}/\sqrt{+2H/m} \tag{48.1}$$

In place of (44) we then have

$$\left.\begin{aligned} [L_i, L_j] &= \epsilon_{ijk}L_k \\ [L_i, J_j] &= \epsilon_{ijk}J_k \\ [J_i, J_j] &= -\epsilon_{ijk}L_k \end{aligned}\right\} \tag{48.2}$$

and are led to construct

$$\mathbb{B} \equiv \begin{pmatrix} 0 & +b_1 & +b_2 & +b_3 \\ b_1 & 0 & -a_3 & +a_2 \\ b_2 & +a_3 & 0 & -a_1 \\ b_3 & -a_2 & +a_1 & 0 \end{pmatrix} = a_1\mathbb{L}_1 + a_2\mathbb{L}_2 + a_3\mathbb{L}_3 + b_1\mathbb{J}_1 + b_2\mathbb{J}_2 + b_3\mathbb{J}_3$$

and to observe that

$$\left.\begin{aligned} [\,\mathbb{L}_i, \mathbb{L}_j] &= \epsilon_{ijk}\mathbb{L}_k \\ [\,\mathbb{L}_i, \,\mathbb{J}_j] &= \epsilon_{ijk}\mathbb{J}_k \\ [\,\mathbb{J}_i\,, \,\mathbb{J}_j] &= -\epsilon_{ijk}\mathbb{L}_k \end{aligned}\right\} \tag{49}$$

But $e^{\mathbb{B}}$ will be recognized to be a <u>Lorentz matrix</u>. The clear implication is that the hyperbolic isoenergetic phase curves $\mathcal{C}_{E>0}$ are interrelated not by elements of $O(4)$ but by elements of the *Lorentz group*! It is curious to find the Lorentz group living right in the middle of one of the most classical of problems, speaking to us of deep things that have nothing at all to do with relativity.

The preceding discussion sprang, as was remarked just above, from the exceptional circumstance that bound orbits in the presence of an attractive $1/r^2$-force all *close upon themselves*. Bertrand's theorem asserts that the same property attaches to one—and only one—alternative force law: the harmonic force. It becomes therefore natural to ask: Can a similar story be told—does "accidental symmetry" arise—also in that case? Indeed it can, and does... as I now demonstrate:

The Hamiltonian $H = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}m\omega^2(x_1^2 + x_2^2)$ of a 2-dimensional isotropic oscillator can be written

$$H = \tfrac{1}{2}\omega(a_1^*a_1 + a_2^*a_2)$$

with

$$\begin{aligned} a_k &\equiv \sqrt{m\omega}\,x_k + ip_k/\sqrt{m\omega} \\ a_k^* &\equiv \sqrt{m\omega}\,x_k - ip_k/\sqrt{m\omega} \end{aligned}$$

Define

$$\begin{aligned} G_1 &\equiv \tfrac{1}{2}\,\omega(a_1^*a_1 - a_2^*a_2) = \tfrac{1}{2m}(p_1^2 - p_2^2) + \tfrac{1}{2}m\omega^2(x_1^2 - x_2^2) \\ G_2 &\equiv \tfrac{1}{2}\,\omega(a_1^*a_2 + a_2^*a_1) = \tfrac{1}{m}\,p_1p_2 + m\omega^2x_1x_2 \\ G_3 &\equiv \tfrac{1}{2i}\omega(a_1^*a_2 - a_2^*a_1) = \omega(x_1p_2 - x_2p_1) \end{aligned}$$

and observe that

$$[H,G_1]=[H,G_2]=[H,G_3]=0$$

G_3-conservation is angular momentum conservation, and is an anticipated reflection of the rotational symmetry of the system. But the other two conservation laws were *not* anticipated (though G_1-conservation is, in retrospect, certainly not hard to understand). Now observe that

$$\begin{aligned}[][G_1,G_2]&=2\omega G_3\\ [G_2,G_3]&=2\omega G_1\\ [G_3,G_1]&=2\omega G_2\end{aligned}$$

and

$$G_1^2+G_2^2+G_3^2=H^2$$

A final notational adjustment

$$\begin{aligned}S_0&\equiv\tfrac{1}{2\omega}H\\ S_i&\equiv\tfrac{1}{2\omega}G_i\end{aligned}$$

places us in position to write

$$[S_i,S_j]=\epsilon_{ijk}S_k$$

$$S_1^2+S_2^2+S_3^2=S_0^2$$

We have again (compare (47)) encountered $O(3)$, manifested this time as the group of canonical transformations that shuffle the isoenergetic curves $\{\mathcal{C}\}_E$ of an isotropic oscillator amongst themselves in 4-dimensional phase space, and that by projection serve to shuffle *centered* ellipses on the $\{x_1,x_2\}$-plane. At this point we have constructed not "apparatus very similar to that which Stokes/Poincaré devised to describe the states of elliptically polarized light" but *precisely* that apparatus, pressed here into alternative physical service. So far as I can determine, $O(4)$ is now *not* hovering in the wings, and certainly we do not have to concern ourselves with unbounded oscillator orbits. What *is* hovering in the wings is the group $SU(2)$ and the associated theory of spinors, as becomes immediately evident when one notices that one can write

$$\begin{aligned}S_0&=\tfrac{1}{4}\begin{pmatrix}a_1^*\\a_2^*\end{pmatrix}^{\mathsf T}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}\\ S_1&=\tfrac{1}{4}\begin{pmatrix}a_1^*\\a_2^*\end{pmatrix}^{\mathsf T}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}\\ S_2&=\tfrac{1}{4}\begin{pmatrix}a_1^*\\a_2^*\end{pmatrix}^{\mathsf T}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}\\ S_3&=\tfrac{1}{4}\begin{pmatrix}a_1^*\\a_2^*\end{pmatrix}^{\mathsf T}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}\end{aligned}$$

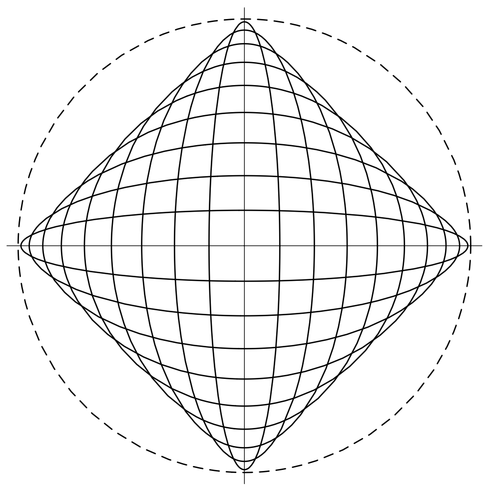

FIGURE 17A: *Isotropic oscillator orbits* (*see again page 20*)

$$r(\theta) = \frac{ab}{\sqrt{a^2 \sin^2(\theta - \delta) + b^2 \cos^2(\theta - \delta)}}$$

of assorted angular momenta $\ell = m\omega ab$, *placed in standard position by setting* $\delta = 0$ *and made isoenergetic by setting* $a^2 + b^2 = 2E/m\omega^2$. *The enveloping circle has radius* $\sqrt{a^2 + b^2}$.

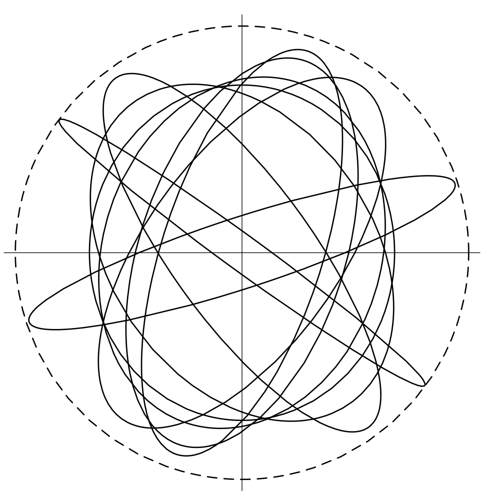

FIGURE 17B: *Representative members of the population* $\{\mathcal{G}\}_E$ *of such orbits that is mapped onto itself under action of generating observables* $\{S_1, S_2, S_3\}$ *that are functionally distinct from, yet algebraically identical to those encountered in connection with the Kepler problem. Here* ℓ *and* δ *have been randomized.*

and that the traceless hermitian 2×2 matrices are **Pauli matrices**, well known to be the generators of the group $SU(2)$ of 2×2 unitary matrices with unit determinant.

Historically, the accidental symmetry issue has been of interest mainly to quantum physicists, who sought to understand why it is that the energy spectra of some systems (most notably the hydrogen atom and the isotropic oscillator) display "accidental degeneracy"—more degeneracy than can be accounted for by obvious symmetry arguments. We have touched here only on one side of the story, the classical side: the more peculiarly quantum side of the story has to do with the fact that the systems known to display accidental degeneracy are systems in which the Schödinger equation can be *separated in more than one coordinate system*.[24]

7. Virial theorem, Bertrand's theorem. I have taken the uncommon step of linking these topics because—despite the physical importance of their applications—both have the feel of "mathematical digressions," and each relates, in its own way, to a *global* property of orbits. Also, I am unlikely to use class time to treat either subject, and exect to feel less guilty about omitting one section than two!

The **virial theorem** was first stated (1870) by Rudolph Clausius (1822–1888), who himself seems to have attached little importance to his invention, though its often almost magical utility was immediately apparent to Maxwell,[25] and it is today a tool very frequently/casually used by atomic & molecular physicists, astrophysicists and in statistical mechanical arguments.[26] Derivations of the virial theorem can be based upon Newtonian[27] or Lagrangian[28] mechanics, but here—because it leads most naturally to certain generalizations—I will employ the apparatus provided by elementary Hamiltonian mechanics.

From Hamilton's equations

$$\begin{aligned}\dot{\boldsymbol{x}} &= +\partial H(\boldsymbol{x},\boldsymbol{p})/\partial\boldsymbol{p}\\ \dot{\boldsymbol{p}} &= +\partial H(\boldsymbol{x},\boldsymbol{p})/\partial\boldsymbol{x}\end{aligned}$$

[24] Readers who wish to pursue the matter might consult H. V. McIntosh, "On accidental degeneracy in classical and quantum mechanics," AJP **27**, 620 (1959) or my own "Classical/quantum theory of 2-dimensional hydrogen" (1999), both of which provide extensive references. For an exhaustive review see McIntosh's "Symmetry & Degeneracy" (2002) at http://delta.cs.cinvestav.mx/m̃cintosh/comun/symm/symm.html.

[25] See his collected *Scientific Papers*, Volume II, page 410.

[26] Consult Google to gain a sense of the remarkable variety of its modern applications.

[27] See, for example, H. Goldstein, *Classical Mechanics* (2nd edition 1980) §3-4. Many applications are listed in Goldstein's index. The 3rd edition (2002) presents the same argument, but omits the list of applications.

[28] See THERMODYNAMICS & STATISTICAL MECHANICS (2002), pages 162–166.

it follows that for any observable $A(\boldsymbol{x},\boldsymbol{p})$ one has

$$\tfrac{d}{dt}A = \sum_{i=1}^{n}\Big\{\frac{\partial A}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial x_i}\frac{\partial A}{\partial p_i}\Big\} \equiv [A,H] \tag{50}$$

Assume the Hamiltonian to have the form

$$H = \tfrac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} + U(\boldsymbol{x})$$

and look (with Clausius) to the case $A = \boldsymbol{p}\cdot\boldsymbol{x}$. Then

$$\begin{aligned}\tfrac{d}{dt}A &= [\boldsymbol{p}\cdot\boldsymbol{x}, \tfrac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p}] + [\boldsymbol{p}\cdot\boldsymbol{x}, U(\boldsymbol{x})]\\ &= \tfrac{1}{m}\boldsymbol{p}\cdot\boldsymbol{p} - \boldsymbol{x}\cdot\boldsymbol{\nabla}U\end{aligned}$$

Writing

$$\overline{a(t)} \equiv \tfrac{1}{\tau}\int_0^\tau a(t)\,dt \equiv \text{time-average of } a(t) \text{ on the indicated interval}$$

we have

$$\overline{\tfrac{d}{dt}A} = \frac{A(\tau)-A(0)}{\tau} = 2\overline{T} + \overline{\boldsymbol{x}\cdot\boldsymbol{\nabla}U}$$

which will vanish if either

- $A(t)$ is periodic with period τ, or
- $A(t)$ is bounded.

Assuming one or the other of those circumstances to prevail, we have

$$\begin{aligned}\overline{T} &= \tfrac{1}{2}\overline{\boldsymbol{x}\cdot\boldsymbol{\nabla}U} \quad : \quad \text{defines what Clausius called the "virial"}\\ &= -\tfrac{1}{2}\overline{\boldsymbol{x}\cdot\boldsymbol{F}}\end{aligned} \tag{51}$$

which is the "virial theorem" in its simplest form.

Suppose it were the case that

$$U(\boldsymbol{x}) \text{ is homogeneous of degree } n$$

Then $\boldsymbol{x}\cdot\boldsymbol{\nabla}U = nU(\boldsymbol{x})$ by Euler's theorem, and the virial theorem becomes

$$\overline{T} = \tfrac{n}{2}\,\overline{U}$$

which pertains more particularly to cases of the familiar type $U = kr^n$. Of course, $E = T + U$ holds at *all* times, so we have

$$E = \big(1+\tfrac{2}{n}\big)\,\overline{T} = \big(1+\tfrac{n}{2}\big)\,\overline{U}$$

giving

$$\begin{aligned}E = 2\overline{T} = 2\overline{U} \quad &: \quad \text{case } n=+2 \text{ (isotropic oscillator)}\\ E = -\overline{T} = \tfrac{1}{2}\overline{U} \quad &: \quad \text{case } n=-1 \text{ (Kepler problem)}\end{aligned}$$

For *circular* orbits both T and U become time-independent (their time-averages become equal to their constant values) and we have

$$E = \left(1 + \tfrac{n}{2}\right)k \cdot (\text{orbital radius})^n$$

In the Keplerian case ($U = -k/r$) one is, on this basis, led directly to the statement (compare (32.1))

$$r_{\text{circular orbit}} = -k/2E$$

which, of course, also follows—but not instantly—from $\boldsymbol{F} = m\ddot{\boldsymbol{x}}$. The essential point, however, is that the relations among E, T and U that hold for circular orbits are shown by the virial theorem to hold *in the time-averaged sense* even for non-circular bound orbits.

Not until 1960 was it noticed[29] that Clausius' virial theorem is but the simplest and most ancient representative of a broad class of such statements, that the argument that led from (50) to (51) is so elemental, has so few moving parts, that it admits of a great many variations, and—more to the point—that *many* of those can be pressed into useful service. Let $A(\boldsymbol{x}, \boldsymbol{p})$ be *any* observable of dynamically bounded variation. Then

$$\overline{[A, \tfrac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p}]} = -\overline{[A, U(\boldsymbol{x})]}$$

and the set of such "hypervirial theorems" can be expanded even further by admitting Hamiltonians of more general design.

In quantum mechanics (50) becomes (in the Heisenberg picture)[30]

$$i\hbar\tfrac{d}{dt}\mathsf{A} = [\mathsf{A}, \mathsf{H}]$$

from which it follows that the motion of $\langle\mathsf{A}\rangle \equiv (\psi|\mathsf{A}|\psi)$—the expected mean of a series of A-measurements, given that the system is in state $|\psi)$—can be described

$$i\hbar\tfrac{d}{dt}\langle\mathsf{A}\rangle = \langle[\mathsf{A}, \mathsf{H}]\rangle$$

If $\langle\mathsf{A}\rangle$ is of bounded variation (or periodic) then after time-averaging we have the "quantum mechanical hypervirial theorem"

$$\overline{\langle[\mathsf{A}, \mathsf{H}]\rangle} = 0$$

which can be particularized in a lot of ways, and from which Hirschfelder and his successors have extracted a remarkable amount of juice. There is, it will be noted, a very close connection between

- **Ehrenfest's theorem**,[31] which speaks about the motion of expected values, and
- **quantum hypervirial theorems**, which speak about time-averaged propeties of such motion.

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[29] J. O. Hirschfelder, "Classical & quantum mechanical hypervirial theorems," J. Chem. Phys. **33**,1462 (1960).

[30] See ADVANCED QUANTUM TOPICS (2000), Chapter 0, page 19.

[31] See Chapter 2, pages 51–60 in the notes just cited.

Joseph Bertrand (1822–1900)—born in the same year as Clausius—was a French mathematician who is today best remembered for "Bertrand's conjecture" (if $n > 1$ then between n and $2n$ can be found at least one prime, and if $n > 3$ the same can be said of the integers between n and $2n-2$: the conjecture was proven by Chebyshev in 1850), but by physicists for **Bertrand's theorem**.[32] A careful—if, it seems to me, disappointingly awkward—proof of Bertrand's theorem is presented as Appendix A in the 2nd edition of Goldstein's *Classical Mechanics* (1980). Goldstein proceeds in Bertrand's pioneering footsteps, or so I believe... and it is in any event in Goldstein's footsteps that we will proceed. We begin with some preparatory remarks concerning <u>circular orbits</u>:

It will be appreciated that in the presence of an attractive central potential $\boldsymbol{F}(\boldsymbol{x}) = -f(r)\hat{\boldsymbol{r}}$ one can have circular orbits of *any* radius. One has simply to "tune the orbital speed" so as to achieve

$$mv^2/r = f(r)$$

or which is the same: to set

$$\ell^2 = mr^3 f(r) \tag{52}$$

If, in particular, the central force derives from a potential of the form $U = kr^n$ (with k taking its sign from n) then $f(r) = nkr^{n-1}$ and we have[33]

$$\ell^2 = nmkr^{n+2}$$

which entails $T = \frac{1}{2mr^2}\ell^2 = \frac{1}{2}nkr^n = (n/2)U$ and so could have been obtained directly from the virial theorem. A circular orbit of radius r_0 will, however, be *stable* (see again FIGURE 5) if an only if the effective potential

$$U_\ell(r) = U(r) + \frac{\ell^2}{2mr^2}$$

is locally *minimal* at r_0: $U_\ell'(r_0) = 0$ and $U_\ell''(r_0) > 0$. In the cases $U = kr^n$ the first condition is readily seen to require

$$r_0^{n+2} = \frac{\ell^2}{mnk}$$

and the second condition therefore entails

$$n > -2$$

---

[32] Bertrand's sister Louise was married to his good friend, Charles Hermite, who was also born in 1822, and survived Bertrand by one year.

[33] Note how odd is the case $n = -2$ from this point of view! ... as it is also from other points of view soon to emerge.

It was remarked already on page 14 that the equation of radial motion can be obtained from the "effective Lagrangian"

$$L = \tfrac{1}{2}m\dot{r}^2 - U_\ell(r)$$

To describe motion in the neighborhood of r_0 we write $r = r_0 + \rho$ and obtain

$$\begin{aligned} L &= \tfrac{1}{2}m\dot{\rho}^2 - \big\{U_\ell(r_0) + U_\ell'(r_0)\rho + \tfrac{1}{2}U_\ell''(r_0)\rho^2 + \cdots\big\} \\ &= \tfrac{1}{2}m\big\{\dot{\rho}^2 - \omega^2\rho^2\big\} + \cdots \end{aligned}$$

The small amplitude solutions will oscillate or explode according as

$$\omega^2 \equiv U_\ell''(r_0)$$

is positive or negative. Looking again to the cases $U = kr^n$ we compute

$$U_\ell''(r_0) = kn(n+2)r_0^{n-2}$$

which is positive if and only $n > -2$. And even when that condition is satisfied, circular orbits with radii different from r_0 are unstable. Stability is the exception, certainly not the rule.

The argument that culminates in Bertrand's theorem has much in common with the argument just rehearsed, the principal difference being that we will concern ourselves—initially in the nearly circular case—not with the temporal oscillations of $r(t)$ but with the angular oscillations of $u(\theta) \equiv 1/r(\theta)$. At (27.1) we had

$$\frac{d^2u}{d\theta^2} + u = J(u) \tag{53}$$

with

$$J(u) = (m/\ell^2)\tfrac{1}{u^2}f(\tfrac{1}{u}) = -(m/\ell^2)\frac{dU(\frac{1}{u})}{du}$$

while at (51) we found that u_0 will refer to a circular orbit (whether stable or—more probably—not) if and only if $\ell^2 = mu_0^{-3}f(1/u_0)$, which we are in position now to express

$$u_0 = J(u_0) \tag{54}$$

Writing $u = u_0 + x$, we now have

$$\frac{d^2x}{d\theta^2} + x = J'(u_0)x + \tfrac{1}{2}J''(u_0)x^2 + \tfrac{1}{6}J'''(u_0)x^3\cdots \tag{55}$$

Leading-order stability—in the present angular sense—requires that

$$\begin{aligned} \beta^2 &\equiv 1 - J'(u_0) \\ &= 1 - (m/\ell^2)\Big\{-2\tfrac{1}{u^3}f\big(\tfrac{1}{u}\big) + \tfrac{1}{u^2}\tfrac{d}{du}f\big(\tfrac{1}{u}\big)\Big\}_{u\to u_0} \\ &= 3 - \Big\{\tfrac{u}{f(1/u)}\tfrac{d}{du}f\big(\tfrac{1}{u}\big)\Big\}_{u\to u_0} \end{aligned} \tag{56}$$

be positive. We then have

$$x(\theta) = a\cos(\beta\theta - \delta) \tag{57}$$

which leads to this very weak conclusion: orbital closure of a perturbed circular orbit requires that β be rational.

That weak conclusion is strengthened, however, by the observation that all $\{E, \ell\}$-assignments that lead to nearly circular orbits must entail the *same* rational number β. Were it otherwise, one would encounter discontinuous orbital-design adjustments as one ranged over that part of parameter space. We observe also that (56) can be written[34]

$$\tfrac{r}{f(r)}\tfrac{d}{dr}f(r) = \tfrac{d\log f}{d\log r} = \beta^2 - 3$$

which integrates at once to give $\log f = (\beta^2 - 3)\log r + \text{constant}$:

$$\begin{aligned} f(r) &= k\,r^{\beta^2-3}\\ &\Downarrow\\ J(u) &= (km/\ell^2)u^{1-\beta^2} \end{aligned} \tag{58}$$

To relax the "nearly circular" assumption we must bring into play the higher-order contributions to (55), and to preserve orbital periodicity/closure we must have[35]

$$x = a_1\cos\beta\theta + \lambda\big\{a_0 + a_2\cos2\beta\theta + a_3\cos3\beta\theta + \cdots\big\} \tag{59}$$

Here λ is a device introduced to identify the terms we expect to be small in the neighborhood of the circle of radius $1/u_0$: once it has done its work it will be set equal to unity. Introducing (59) into (55)—which now reads

$$\frac{d^2x}{d\theta^2} + \beta^2 x = \tfrac{1}{2}J''(u_0)x^2 + \tfrac{1}{6}J'''(u_0)x^3 + \cdots$$

—we execute the command `Series[`*expression*`,`$\{\lambda,0,1\}$`]`, then set $\lambda = 1$, then command `TrigReduce[`*expression*`]` to turn $\cos^2$ and $\cos^3$ terms into their multiple-angle equivalents, and are led by these manipulations to write

$$\begin{aligned}
\beta^2a_0 + 0 - 3\beta^2a_2\cos2\beta\theta - 8\beta^2a_3\cos3\beta\theta + \cdots&\\
= \quad&\big\{\tfrac{1}{4}a_1^2J'' + a_1^2\big(\tfrac{1}{4}a_0 + \tfrac{1}{8}a_2\big)J''' + \cdots\big\}\\
&+\big\{a_1\big(a_0 + \tfrac{1}{2}a_2\big)J'' + \tfrac{1}{8}a_1^2\big(a_1 + a_3\big)J''' + \cdots\big\}\cos\beta\theta\\
&+\big\{\tfrac{1}{4}a_1\big(a_1 + 2a_3\big)J'' + \tfrac{1}{4}a_1^2\big(a_0 + a_2\big)J''' + \cdots\big\}\cos2\beta\theta\\
&+\big\{\tfrac{1}{2}a_1a_2J'' + a_1^2\big(\tfrac{1}{24}a_1 + \tfrac{1}{4}a_3\big)J''' + \cdots\big\}\cos3\beta\theta + \cdots
\end{aligned}$$

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[34] Use $u\frac{d}{du} = -r\frac{d}{dr}$, which follows directly from $u = 1/r$.

[35] It is without loss of generality that we have dropped the δ-term from (57), for it can always be absorbed into a redefinition of the point from which we measure θ.

and to conclude that

$$\begin{aligned}
a_0 &= \tfrac{1}{\beta^2}\big\{\tfrac{1}{4}a_1^2J'' + a_1^2\big(\tfrac{1}{4}a_0 + \tfrac{1}{8}a_2\big)J''' + \cdots\big\}\\
0 &= \big\{a_1\big(a_0 + \tfrac{1}{2}a_2\big)J'' + \tfrac{1}{8}a_1^2\big(a_1 + a_3\big)J''' + \cdots\big\}\\
a_2 &= -\tfrac{1}{3\beta^2}\big\{\tfrac{1}{4}a_1\big(a_1 + 2a_3\big)J'' + \tfrac{1}{4}a_1^2\big(a_0 + a_2\big)J''' + \cdots\big\}\\
a_3 &= -\tfrac{1}{8\beta^2}\big\{\tfrac{1}{2}a_1a_2J'' + a_1^2\big(\tfrac{1}{24}a_1 + \tfrac{1}{4}a_3\big)J''' + \cdots\big\}
\end{aligned}$$

From the specialized structure (58) of $J(u)$ that has been forced upon us it follows that

$$\begin{aligned}
J''(u) &= -\beta^2(1-\beta^2)u^{-2}J(u)\\
J'''(u) &= (1+\beta^2)\beta^2(1-\beta^2)u^{-3}J(u)
\end{aligned}$$

so at $u_0 = J(u_0)$ we have

$$\begin{aligned}
J'' &= -\beta^2(1-\beta^2)/u_0\\
&\equiv \beta^2 J_2/u_0\\
J''' &= (1+\beta^2)\beta^2(1-\beta^2)/u_0^2\\
&\equiv \beta^2 J_3/u_0^2
\end{aligned}$$

giving

$$\begin{aligned}
\tfrac{a_0}{a_1} &= \big\{\tfrac{1}{4}\tfrac{a_1}{u_0}J_2 + \tfrac{a_1}{u_0}\big(\tfrac{1}{4}\tfrac{a_0}{u_0} + \tfrac{1}{8}\tfrac{a_2}{u_0}\big)J_3 + \cdots\big\}\\
0 &= a_1\big\{\big(\tfrac{a_0}{u_0} + \tfrac{1}{2}\tfrac{a_2}{u_0}\big)J_2 + \tfrac{1}{8}\tfrac{a_1}{u_0}\big(\tfrac{a_1}{u_0} + \tfrac{a_3}{u_0}\big)J_3 + \cdots\big\}\\
\tfrac{a_2}{a_1} &= -\tfrac{1}{3}\big\{\tfrac{1}{4}\big(\tfrac{a_1}{u_0} + 2\tfrac{a_3}{u_0}\big)J_2 + \tfrac{1}{4}\tfrac{a_1}{u_0}\big(\tfrac{a_0}{u_0} + \tfrac{a_2}{u_0}\big)J_3 + \cdots\big\}\\
\tfrac{a_3}{a_1} &= -\tfrac{1}{8}\big\{\tfrac{1}{2}\tfrac{a_2}{u_0}J_2 + \tfrac{a_1}{u_0}\big(\tfrac{1}{24}\tfrac{a_1}{u_0} + \tfrac{1}{4}\tfrac{a_3}{u_0}\big)J_3 + \cdots\big\}
\end{aligned}$$

which we now rewrite in such a way as to expose implications of our presumption that the ratios a_0/u_0 and a_1/u_0 are small:

$$\begin{aligned}
\tfrac{a_0}{a_1} &= \big\{\tfrac{1}{4}\tfrac{a_1}{u_0}J_2 + \tfrac{a_1}{u_0}\big(\tfrac{1}{4}\tfrac{a_0}{u_0} + \tfrac{1}{8}\tfrac{a_1}{u_0}\tfrac{a_2}{a_1}\big)J_3 + \cdots\big\}\\
0 &= a_1\big\{\big(\tfrac{a_1}{u_0}\tfrac{a_0}{a_1} + \tfrac{1}{2}\tfrac{a_1}{u_0}\tfrac{a_2}{a_1}\big)J_2 + \tfrac{1}{8}\tfrac{a_1}{u_0}\big(\tfrac{a_1}{u_0} + \tfrac{a_1}{u_0}\tfrac{a_3}{a_1}\big)J_3 + \cdots\big\}\\
\tfrac{a_2}{a_1} &= -\tfrac{1}{3}\big\{\tfrac{1}{4}\big(\tfrac{a_1}{u_0} + 2\tfrac{a_1}{u_0}\tfrac{a_3}{a_1}\big)J_2 + \tfrac{1}{4}\tfrac{a_1}{u_0}\big(\tfrac{a_0}{u_0} + \tfrac{a_1}{u_0}\tfrac{a_2}{a_1}\big)J_3 + \cdots\big\}\\
\tfrac{a_3}{a_1} &= -\tfrac{1}{8}\big\{\tfrac{a_1}{u_0}\tfrac{a_2}{a_1}J_2 + \tfrac{a_1}{u_0}\big(\tfrac{1}{24}\tfrac{a_1}{u_0} + \tfrac{1}{4}\tfrac{a_1}{u_0}\tfrac{a_3}{a_1}\big)J_3 + \cdots\big\}
\end{aligned}$$

The implication of interest is that a_3/a_1 is "small-small" (of order $(a_1/u_0)^2$). In leading order we therefore have

$$\begin{aligned}
\tfrac{a_0}{a_1} &= \tfrac{1}{4}\tfrac{a_1}{u_0}J_2\\
0 &= \big\{\tfrac{a_1}{u_0}\big(\tfrac{a_0}{a_1} + \tfrac{1}{2}\tfrac{a_2}{a_1}\big)J_2 + \tfrac{1}{8}\tfrac{a_1}{u_0}\tfrac{a_1}{u_0}J_3 + \cdots\big\}\\
\tfrac{a_2}{a_1} &= -\tfrac{1}{12}\tfrac{a_1}{u_0}J_2
\end{aligned}$$

Feeding the first and third of these equations into the second, we obtain

$$\begin{aligned}
0 = \tfrac{1}{24}\big(\tfrac{a_1}{u_0}\big)^2\big[5J_2^2 + 3J_3\big] &= \tfrac{1}{24}\big(\tfrac{a_1}{u_0}\big)^2\big[5(1-\beta^2)^2 + 3(1+\beta^2)(1-\beta^2)\big]\\
&= \tfrac{1}{12}\big(\tfrac{a_1}{u_0}\big)^2\big[\beta^2 - 5\beta^2 + 4\big]
\end{aligned}$$

and are brought thus (with Bertrand) to the conclusion that the β^2 in

$$f(r) = k\,r^{\,\beta^2-3}$$

is constrained to satisfy

$$(\beta^2-4)(\beta^2-1) = 0$$

The attractive central forces that give rise to invariably closed bounded orbits are two—and only two—in number:

$$\begin{aligned} f(r) &= -kr^{+1} &&: \quad \text{HARMONIC} \\ f(r) &= +kr^{-2} &&: \quad \text{KEPLERIAN} \end{aligned}$$

The preceding argument is in some technical sense "elementary," and certainly it is, at several points, quite ingenious... if (in my view) not entirely convincing. It seems to me to be fundamentally misguided to appeal to a rough-&-ready *ad hoc* perturbation theory to establish a global result, and would be surprising if such a strikingly clean and simple result—such a pretty vista—can be approached only by such a rocky trail. It is my intuitive sense that an elegant two-line argument—global from start to finish—awaits discovery. J. V. José & E. J. Saletan, in §2.3.3 of their excellent *Classical Dynamics: A Contemporary Approach* (1998), provide a sketch of an alternative argument devised by Arnol'd,[36] but it does not seem to me to be much of an improvement.

8. Double separation of the Hamilton-Jacobi equation. Bertrand's theorem identifies the harmonic and Keplerian central forces as "special" in an orbital regard that makes good sense classically, but that supports no direct quantum mechanical interpretation. Those same two "special cases" were found in §7 to be linked also in another regard: both display "accidental/hidden symmetries" that lead to unanticipated/non-obvious conservation laws, and those *do* find a place in the associated quantum theories. The harmonic and Keplerian central force systems are, as it happens, "special" in yet a third regard: both are—uniquely, so far as anyone knows—"multiply separable," where the term refers

- classically to separation of the Hamilton-Jacobi equation in more than one coordinate system, and
- in quantum mechanics to separation of the Schrödinger equation.

It is to that third part of the story that we now turn. It is widely assumed that the three parts are interrelated, symptoms of something deep... though the identity of that "something" lives mainly in vague folklore and not yet in the world of clearly stated mathematical fact (see again the McIntosh papers[24] cited previously).

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[36] V. I. Arnol'd, *Mathematical Methods of Classical Mechanics* (2nd edition 1988). José & Saletan cite also a passage in E. T. Whittaker's *Analytical Mechanics* (4th edition 1937), but I don't think they actually read the §108 to which they refer: though entitled "Bertrand's theorem," it treats quite a different mechanical proposition.

ISOTROPIC HARMONIC OSCILLATOR

Separation in Cartesian coordinates: From $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}m\omega^2(x^2 + y^2)$ we obtain $p_x = m\dot{x}$, $p_y = m\dot{y}$ whence

$$\begin{aligned} H(x, y, p_x, p_y) &= p_x\dot{x} + p_y\dot{y} - L(x, y, \dot{x}, \dot{y}) \\ &= \tfrac{1}{2m}\big[p_x^2 + p_y^2\big] + \tfrac{1}{2}m\omega^2(x^2 + y^2) \end{aligned}$$

and the time-independent Hamilton-Jacobi equation becomes

$$\tfrac{1}{2m}\big[(\tfrac{\partial S}{\partial x})^2 + (\tfrac{\partial S}{\partial y})^2\big] + \tfrac{1}{2}m\omega^2(x^2 + y^2) = E$$

Assume $S(x, y)$ to have the form

$$S(x, y) = X(x) + Y(y)$$

and obtain the separated equations

$$\left.\begin{aligned} \tfrac{1}{2m}(\tfrac{dX}{dx})^2 + \tfrac{1}{2}m\omega^2x^2 &= \tfrac{1}{2}E + \lambda \\ \tfrac{1}{2m}(\tfrac{dY}{dy})^2 + \tfrac{1}{2}m\omega^2y^2 &= \tfrac{1}{2}E - \lambda \end{aligned}\right\} \tag{60.1}$$

where λ is a separation constant (as was E). The initial PDE has been resolved into an uncoupled pair of ODEs.

Separation in polar coordinates: Write

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Then $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}m\omega^2r^2$ gives $p_r = m\dot{r}$, $p_\theta = mr^2\dot{\theta}$ whence

$$\begin{aligned} H(r, \theta, p_r, p_\theta) &= p_r\dot{r} + p_\theta\dot{\theta} - L(r, \theta, \dot{r}, \dot{\theta}) \\ &= \tfrac{1}{2m}p_r^2 + \tfrac{1}{2mr^2}p_\theta^2 + \tfrac{1}{2}m\omega^2r^2 \end{aligned}$$

and the H-J equation becomes

$$\tfrac{1}{2m}(\tfrac{\partial S}{\partial r})^2 + \tfrac{1}{2mr^2}(\tfrac{\partial S}{\partial \theta})^2 + \tfrac{1}{2}m\omega^2r^2 = E$$

Assume $S(r, \theta)$ to have the form

$$S(r, \theta) = R(r) + T(\theta)$$

and obtain the separated equations

$$\left.\begin{aligned} \tfrac{1}{2m}(\tfrac{dT}{d\theta})^2 - \lambda &= 0 \\ \tfrac{1}{2m}(\tfrac{dR}{dr})^2 + \tfrac{1}{2}m\omega^2r^2 + \tfrac{1}{r^2}\lambda &= E \end{aligned}\right\} \tag{60.2}$$

Separation in "alternate polar coordinates": Fundamentally equivalent to, but for many applications more attractive than, the familiar polar coordinate system is the "alternate polar system" defined

$$\left.\begin{aligned} x &= ae^s\cos\theta \\ y &= ae^s\sin\theta \end{aligned}\right\} \quad : \quad a \text{ has arbitrary value, dimensions of length}$$

From $L = \frac{1}{2}ma^2e^{2s}(\dot{s}^2+\dot{\theta}^2) - \frac{1}{2}m\omega^2a^2e^{2s}$ we obtain $p_s = ma^2e^{2s}\dot{s}$, $p_\theta = ma^2e^{2s}\dot{\theta}$ whence

$$H(s,\theta,p_s,p_\theta) = \tfrac{1}{2ma^2}e^{-2s}\big[p_s^2+p_\theta^2\big] + \tfrac{1}{2}ma^2\omega^2e^{2s}$$

and the H-J equation becomes

$$\tfrac{1}{2ma^2}e^{-2s}\big[(\tfrac{\partial S}{\partial s})^2 + (\tfrac{\partial S}{\partial\theta})^2\big] + \tfrac{1}{2}ma^2\omega^2e^{2s} = E$$

Assume $S(s,\theta)$ to have the form

$$S(s,\theta) = S(s) + T(\theta)$$

and obtain the separated equations

$$\left.\begin{aligned} \tfrac{1}{2ma^2}(\tfrac{dT}{d\theta})^2 &= +\lambda \\ \tfrac{1}{2ma^2}(\tfrac{\partial S}{\partial s})^2 + \tfrac{1}{2}ma^2\omega^2e^{4s} - Ee^{2s} &= -\lambda \end{aligned}\right\} \tag{60.3}$$

2-DIMENSIONAL KEPLER PROBLEM

Separation in polar coordinates: An direct implication of preceding discussion is that in the present instance

$$H(r,\theta,p_r,p_\theta) = \tfrac{1}{2m}p_r^2 + \tfrac{1}{2mr^2}p_\theta^2 - \tfrac{k}{r}$$

so the H-J equation reads

$$\tfrac{1}{2m}(\tfrac{\partial S}{\partial r})^2 + \tfrac{1}{2mr^2}(\tfrac{\partial S}{\partial\theta})^2 - \tfrac{k}{r} = E$$

Assume $S(r,\theta)$ to have the form

$$S(r,\theta) = R(r) + T(\theta)$$

and obtain the separated equations

$$\left.\begin{aligned} \tfrac{1}{2m}(\tfrac{dT}{d\theta})^2 - \lambda &= 0 \\ \tfrac{1}{2m}(\tfrac{dR}{dr})^2 - \tfrac{k}{r} + \tfrac{1}{r^2}\lambda &= E \end{aligned}\right\} \tag{61.1}$$

Separation in alternate polar coordinates: Again, it follows at once from recent work that

$$H(s,\theta,p_s,p_\theta) = \tfrac{1}{2ma^2}e^{-2s}\big[p_s^2+p_\theta^2\big] - \tfrac{k}{a}e^{-s}$$

so we have

$$\tfrac{1}{2ma^2}e^{-2s}\big[(\tfrac{\partial S}{\partial s})^2+(\tfrac{\partial S}{\partial\theta})^2\big]-\tfrac{k}{a}e^{-s}=E$$

Assume $S(s,\theta)$ to have the form

$$S(s,\theta)=S(s)+T(\theta)$$

and obtain the separated equations

$$\left.\begin{aligned}\tfrac{1}{2ma^2}(\tfrac{dT}{d\theta})^2&=+\lambda\\ \tfrac{1}{2ma^2}(\tfrac{\partial S}{\partial s})^2-\tfrac{k}{a}e^{s}-Ee^{2s}&=-\lambda\end{aligned}\right\}\tag{61.2}$$

Separation in confocal parabolic coordinates: In Cartesian coordinates we have

$$L=\tfrac{1}{2}m(\dot x^2+\dot y^2)+\frac{k}{\sqrt{x^2+y^2}}$$

The coordinate system of present interest (see FIGURE 18) arises when one writes[37]

$$\begin{aligned}x&=\tfrac{1}{2}(\mu^2-\nu^2)\\ y&=\mu\nu\end{aligned}$$

Straightforward calculation supplies

$$L=\tfrac{1}{2}m(\mu^2+\nu^2)(\dot\mu^2+\dot\nu^2)+\frac{k}{\mu^2+\nu^2}$$

whence $p_\mu=m(\mu^2+\nu^2)\dot\mu$ and $p_\nu=m(\mu^2+\nu^2)\dot\nu$, from which we obtain

$$H(\mu,\nu,p_\mu,p_\nu)=\tfrac{1}{\mu^2+\nu^2}\Big\{\tfrac{1}{2m}\big[p_\mu^2+p_\nu^2\big]-2k\Big\}$$

The H-J equation therefore reads

$$\tfrac{1}{2m}\big[(\tfrac{\partial S}{\partial\mu})^2+(\tfrac{\partial S}{\partial\nu})^2\big]-2k=(\mu^2+\nu^2)E$$

Assuming $S(\mu,\nu)$ to have the form

$$S(\mu,\nu)=M(\mu)+N(\nu)$$

we obtain separated equations

$$\left.\begin{aligned}\tfrac{1}{2m}(\tfrac{dM}{d\mu})^2-\mu^2E&=k+\lambda\\ \tfrac{1}{2m}(\tfrac{dN}{d\nu})^2-\nu^2E&=k-\lambda\end{aligned}\right\}\tag{61.3}$$

that are notable for their elegant symmetry. It will be noted that when $E<0$ these resemble an equation basic to the theory of oscillators.

[37] See P. Moon & D. E. Spencer, *Field Theory Handbook* (1961), page 21.

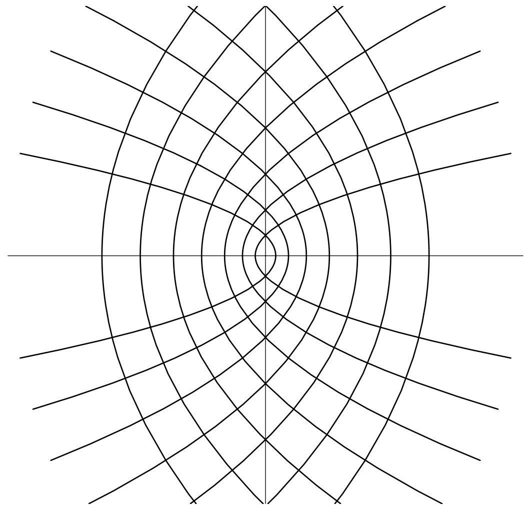

FIGURE 18: *Confocal parabolic coordinate system: x runs $\rightarrow$, y runs $\uparrow$, curves of constant ν open to the right, curves of constant μ open to the left. Confocal parabolic coordinates are particularly well adapted to discussion of all aspects of the Kepler problem, both classically and quantum mechanically.*

9. Euler's "two centers problem".

10. Kepler problem in action-angle variables.

11. Three-body problem.

12. Perturbation theory.

13. Ballistics.

14. Scattering by a central force.

15. Higher-dimensional analog of the central force problem.